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Structure of equicontinuous foliated spaces

(Estructura de espacios foliados equicontinuos)

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CHAPTER 1

Abstract

The main goal of this thesis is to prove the structural theorems of Molino's theory of Riemannian foliations in the topological context; in other words, for compact equicontinuous foliated spaces. This is achieved for the case where the leaves are dense (minimal foliated space), describing such foliated space in terms of a foliated space whose transverse dynamics is given by local left translations in a local group G (a G-foliated space). Then this description is used to study the growth of the leaves in the spirit of Carriére and Breuillard-Gelander.

To be more precise, let X be a compact Polish minimal foliated space. Its transverse dynamics is given by the maps between local transversals obtained by sliding them along the leaves. They are called holonomy transformations and form the holonomy pseudogroup $\mathcal{H}$ of X. If the local transversals are slided along loops in a leaf L, then we get a representation of $\pi_1(L)$ in a group of germs of holonomy transformations. This gives rise to the holonomy group of L and the holonomy cover $\widetilde{L} \to L$.

In the case of a foliation on a compact manifold, the growth of the leaves and their holonomy covers is defined with the metric induced by a Riemannian metric on the ambient manifold; it depends only on the foliation by the compactness of the ambient manifold. For compact foliated spaces, it is still possible to define their growth by using "coarse metrics" on the leaves and their holonomy covers, given by the plaques of a finite foliated atlas.

Assume that X is equicontinuous in the sense that some set S of generators of $\mathcal{H}$, closed by the operations of composition and inversion, is uniformly equicontinuous; this S is called a pseudo$*$group. Then Álvarez and Candel have proved that $\mathcal{H}$ has a closure $\overline{\mathcal{H}}$, which is the pseudogroup of maps that can be locally obtained as limits of maps in $\mathcal{H}$ with respect to the compact-open topology. Suppose also that $\overline{\mathcal{H}}$ is strongly quasi-analytic in the sense that there is a generating pseudo$*$group $\overline{S} \subset \overline{\mathcal{H}}$ such that any map in $\overline{S}$ is the identity map on its domain if it is the identity on some non-empty open subset. Then our first main theorem states that there is a compact Polish minimal G-foliated space $\widehat{X}_0$, for some local group G, and a foliated projection $\hat{\pi}_0 : \widehat{X}_0 \to X$, whose fibers are homeomorphic one another, and whose restrictions to the leaves are the holonomy covers of the leaves of X. This G is called the structural local group of X. Our second main theorem

states that, with the same hypotheses, one of the following properties hold: G can be “approximated” by nilpotent local Lie groups, or the holonomy covers of all leaves have exponential growth.

Chapter 2
Introduction

This work is about equicontinuous foliated spaces, considered as generalizations of the Riemannian foliations introduced by Reinhart [40]. Specially, we consider compact equicontinous spaces that are minimal in the sense that their leaves are dense.

It was pointed out by Ghys in [34, Appendix E] (see also Kellum's paper [28]) that equicontinuous foliated spaces should be considered as the "topological Riemannian foliations", and therefore many of the results about Riemannian foliations should have versions for equicontinuous foliated spaces. Some steps in this direction were given by Álvarez and Candel [4, 5], showing that, under reasonable conditions on compact equicontinuous foliated spaces, their leaf closures are minimal foliated spaces, their generic leaves are quasi-isometric to each other, and their holonomy pseudogroup has a closure, like in the case of Riemannian foliations. In the same direction, Matsumoto [30] proved that any minimal equicontinuous foliated space has a non-trivial transverse invariant measure, which is unique up to products by constants. The magnitude of the generalization from Riemannian foliations to equicontinuous foliated spaces was made precise by Álvarez and Candel [5] (see also Tarquini's paper [43]), giving a topological description of Riemannian foliations within the class of equicontinuous foliated spaces.

Most of the known properties of Riemannian foliations follow from a description due to Molino [33, 34]. However, so far, there was no version of Molino's description for foliated spaces—the indicated properties of equicontinuous foliated spaces were obtained by other means. The goal of our work is to develop such a version of Molino's theory, and use it to study the growth of their leaves, in the same spirit of the study of the growth of Riemannian foliations by Carrière [13] (see also the recent paper [11] by Breuillard-Gelander). To understand our results better, let us briefly recall Molino's theory—a more thorough description of that theory is given in Appendix A.

Molino's theory for Riemannian foliations

Recall that a (smooth) *foliation* $\mathcal{F}$ of *codimension* q on a manifold M is a partition of M into injectively immersed connected submanifolds (*leaves*), which can be locally described as the fibers of local submersions onto q-manifolds. These submersions and their domains are said to be *distinguished*, and their images are called *local quotients*. The changes of distinguished submersions are given by diffeomorphisms between open subsets of the local quotients, which are called *elementary holonomy transformations*. A foliation is called *minimal* if the leaves are dense. A map between foliated manifolds is called *foliated* if it maps leaves to leaves.

By using chains of consecutive distinguished open sets along loops in a leaf L, and composing the corresponding elementary holonomy transformations, we get a representation of $\pi_1(L)$ in a group of germs of those compositions, which is called the *holonomy representation* of L. Its image is called the *holonomy group* of L, and its kernel equals the image of the homomorphism $\pi_1(\widetilde{L}) \to \pi_1(L)$ induced by a unique regular cover $\widetilde{L} \to L$, which is called *holonomy cover*. For a general foliation on a second countable manifold, there is a dense G_δ saturated subset whose leaves have trivial holonomy groups [26, 12]; thus any statement about the holonomy covers of he leaves can be simplified as a statement about the generic leaves if desired.

Let $T\mathcal{F} \subset TM$ denote the vector subbundle of vectors tangent to the leaves. Then $N\mathcal{F} = TM/T\mathcal{F}$ is called the *normal bundle* of $\mathcal{F}$, and its sections *normal vector fields*. There is a natural flat leafwise partial connection on $N\mathcal{F}$ so that a local normal vector field is leafwise parallel if it is locally projectable by the distinguished submersions; terms like "leafwise flat," "leafwise parallel" and "leafwise horizonta" will refer to this partial connection. It is said that $\mathcal{F}$ is:

Riemannian if there is a leafwise parallel Riemannian structure on $N\mathcal{F}$;

transitive if the group of its foliated diffeomorphisms acts transitively on M;

transversely parallelizable (TP) if there is a leafwise parallel global frame of $N\mathcal{F}$, called *transverse parallelism*; and a

Lie foliation if moreover the transverse parallelism is a basis of a Lie algebra with the operation induced by the vector field bracket.

These conditions are successively stronger. Intuitively, a foliation is Riemannian when their leaves do not get too close or too far by traveling along them.

Molino's theory describes Riemannian foliations on compact manifolds in terms of minimal Lie foliations, and using TP foliations as an intermediate step:

1st step: If $\mathcal{F}$ is Riemannian and M compact, then there is an $\mathrm{O}(q)$-principal bundle, $\hat{\pi} : \widehat{M} \to M$, with an $\mathrm{O}(q)$-invariant TP foliation, $\widehat{\mathcal{F}}$, such that

$\widehat{\pi}$ is a foliated map whose restrictions to the leaves are the holonomy covers of the leaves of $\mathcal{F}$.

2nd step: If $\mathcal{F}$ is TP and M compact, then there is a fiber bundle $\pi : M \to W$ whose fibers are the leaf closures of $\mathcal{F}$, and the restriction of $\mathcal{F}$ to each fiber is a Lie foliation.

The proofs of these statements use strongly the differenttial structure of $\mathcal{F}$. In the 1st step, $\hat{\pi} : \widehat{M} \to M$ is the $\mathrm{O}(q)$-principal bundle of orthonormal frames for some leafwise parallel metric on $N\mathcal{F}$, and $\widehat{\mathcal{F}}$ is given by the corresponding flat leafwise horizontal distribution. Then $\widehat{\mathcal{F}}$ is TP by an adaptation of a standard argument. In the 2nd step, foliated flows are used to produce fiber bundle trivializations whose fibers are the leaf closures; this works because there are foliated flows in any transverse direction since $\mathcal{F}$ is TP.

When $\mathcal{F}$ is minimal, we get the following:

Minimal case: If $\mathcal{F}$ is minimal and Riemannian, and M is compact, then, for some closed subgroup $H \subset \mathrm{O}(q)$, there is an H-principal bundle, $\hat{\pi}_0 : \widehat{M_0} \to M$, with an H-invariant minimal Lie foliation, $\widehat{\mathcal{F}}_0$, such that $\widehat{\pi}_0$ is a foliated map whose restrictions to the leaves are the holonomy covers of the leaves of $\mathcal{F}$.

This follows from the combination of both steps by observing that any leaf closure $\widehat{M_0}$ of $\widehat{\mathcal{F}}$ is a principal subbundle of $\hat{\pi} : \widehat{M} \to M$.

A useful description of Lie foliations was also given by Fédida [17, 18], but it will not be considered here.

Holonomy of Riemannian foliations

A *pseudogroup* is a maximal collection of local transformations of a space, which contains the identity map, and is closed under the operations of composition, inversion, restriction and combination. It can be considered as a generalized dynamical system, and all basic dynamical concepts have pseudogroup versions. They are relevant in foliation theory because the elementary holonomy transformations generate a pseudogroup which describes the transverse dynamics of $\mathcal{F}$; it is called the *holonomy pseudogroup* and its elements *holonomy transformations*. Such a pseudogroup is well determined up to certain *equivalence* of pseudogroups introduced by Haefliger [22, 23]. We may say that $\mathcal{F}$ is *transversely modeled* by some class of local transformations if its holonomy pseudogroup can be generated by that type of local transformations. Riemannian, TP and Lie foliations can be respectively characterized by being transversely modeled by

- local isometries of some Riemannian manifold;
- local diffeomorphisms of a parallelizable manifold preserving the parallelism; and
- left translations of a Lie group.

Thus Riemannian foliations are the transversely rigid ones, and TP foliations have a stronger type of transverse rigidity.

When the ambient manifold M is compact, Haefliger [25] has observed that the holonomy pseudogroup $\mathcal{H}$ of $\mathcal{F}$ satisfies the following property:

Compact generation: There is some relatively compact open subset $U \subset T$, which meets all $\mathcal{H}$-orbits, and there is a finite number of generators $h_1, \dots, h_k$ of the restriction $\mathcal{H}|_U$ such that each h_i has an extension $\tilde{h} \in \mathcal{H}$ with $\overline{\operatorname{dom} h} \subset \operatorname{dom} \tilde{h}$.

If moreover $\mathcal{F}$ is Riemannian, then Haefliger [23, 25] has also strongly used the following properties of $\mathcal{H}$:

Completeness: For all $x, y \in T$, there are open neighborhoods, V of x and W of y, such that, for all $h \in \mathcal{H}$ and $z \in V \cap \operatorname{dom} h$ with $h(z) \in W$, there is some $\tilde{h} \in \mathcal{H}$ such that $\operatorname{dom} \tilde{h} = V$ and $\tilde{h} = h$ around z.

Closure: Let $J^1(T)$ be the space of 1-jets of local transformations of T, and let $j^1(\mathcal{H}) \subset J^1(T)$ the subset given by 1-jets of maps in $\mathcal{H}$. Then the closure $\overline{j^1(\mathcal{H})}$ in $J^1(T)$ is the set of 1-jets of maps in a pseudogroup $\overline{\mathcal{H}}$ of local isometries of T, called the *closure* of $\mathcal{H}$, whose orbits are the closures of the $\mathcal{H}$-orbits.

Quasi-analyticity: If some $h \in \mathcal{H}$ is the identity on some open set O with $\overline{O} \subset \operatorname{dom} h$, then h is the identity on some neighborhood of $\overline{O}$.

Quasi-analyticity holds because the differential of an isometry at some point determines the map on a neighborhood. Thus it also holds for $\overline{\mathcal{H}}$.

For a compactly generated pseudogroup $\mathcal{H}$ of local isometries of a Riemannian manifold T, Salem has given a version of Molino's theory ([42] and [34, Appendix D]; see also [8]). In particular, in the minimal case, it turns out that there is a Lie group G, a compact subgroup $K \subset G$ and a dense finitely generated subgroup $\Gamma \subset G$ such that $\mathcal{H}$ is equivalent to the pseudogroup generated by the action of Γ on the homogeneous space G/K (this was also observed by Haefliger [23]).

Growth of Riemannian foliations

Molino's theory has many consequences for a Riemannian foliation $\mathcal{F}$ on a compact manifold M: classification in particular cases, growth, cohomology, tautness and tenseness, and global analysis. In all of them, Molino's theory is used to reduce the study to the case of Lie foliations with dense leaves, where it usually becomes a problem of Lie theory. A list of references about all applications would be too long. We concentrate on the consequences about the growth of the leaves and their holonomy covers, which refers to their growth as Riemannian manifolds with the metrics induced by any metric on M; this growth depends only on $\mathcal{F}$ by the compactness of M. This study was begun by Carrière [13], and recently continued by Breuillard-Gelander, as a consequence of their study of a topological Tits alternative [11]. Their results state the following, where $\mathfrak{g}$ is the structural Lie algebra of $\mathcal{F}$:

Carrière's theorem: The holonomy covers of the leaves are:

- Følner if and only if $\mathfrak{g}$ is solvable; and
- of polynomial growth if and only if $\mathfrak{g}$ is nilpotent.

Moreover, in the second case, the degree of their polynomial growth is bounded by the nilpotence degree of $\mathfrak{g}$.

Breuillard-Gelander's theorem: The growth of the holonomy covers of the leaves is either polynomial or exponential.

Equicontinuous foliated spaces

A *foliated space* $X \equiv (X, \mathcal{F})$ is a topological space X equipped with partition $\mathcal{F}$ into connected manifolds (*leaves*), which can be locally described by the fibers of topological submersions. It will be assumed that X is locally compact and Polish. A foliated space should be considered as a "topological foliation". In this sense, all topological notions of foliations have obvious versions for foliated spaces. In particular, the *holonomy pseudogroup* $\mathcal{H}$ of X is defined on a locally compact Polish space T. Many results about foliations also have straightforward generalizations; for example, the leaves with trivial holonomy groups form a dense G_δ set, and $\mathcal{H}$ is compactly generated if X is compact. Even leafwise differential concepts are easy to extend. However this task may be difficult or impossible for transverse differential concepts. For instance, the normal bundle of a foliated space does not make any sense in general: it would be the tangent bundle of a topological space in the case of a space foliated by points. Thus the concept of Riemannian foliation cannot be extended by using the normal bundle; instead, this can be done via the holonomy pseudogroup as follows.

The transverse rigidity of a Riemannian foliation can be translated to the foliated space X by requiring (uniform) equicontinuity of $\mathcal{H}$; in fact, the equicontinuity condition is not compatible with combinations of maps; thus equicontinuity is indeed required for some generating subset $S \subset \mathcal{H}$ which is closed by the operations of composition and inversion; such an S is called a *pseudo∗group* with the terminology of Matsumoto [30]. This gives rise to the concept of *equicontinuous* foliated space, which should be regarded as the topological version of a Riemannian foliation.

Like in the Riemannian foliation case, Álvarez and Candel [4] have proved that, if the foliated space X is compact and equicontinuous, its leaf closures are minimal foliated spaces, and its holonomy pseudogroup $\mathcal{H}$ is complete and has a closure $\overline{\mathcal{H}}$. With this generality, $\overline{\mathcal{H}}$ cannot be defined by using 1-jets, of course; instead, $\overline{\mathcal{H}}$ consists of the maps that locally are limits of maps in $\mathcal{H}$ with the compact-open topology; this method works well because $\mathcal{H}$ is complete.

In the topological setting, the quasi-analyticity of $\mathcal{H}$ (and $\overline{\mathcal{H}}$) does not follow from the equicontinuity assumption. Thus quasi-analyticity will be required as an additional assumption when needed. Indeed, it does not work well enough when T is not locally connected. So we use a property called *strong quasi-analyticity*, defined by the existence of a pseudo∗group S, generating $\mathcal{H}$, such that any map in S is the identity on its domain if it is the identity on some non-empty open subset; this property is stronger that quasi-analyticity just when T is not locally connected.

Transitive and Lie foliations have the following obvious topological versions. Given a local group G, it is said that the foliated space X is:

homogeneous if the group of its foliated transformations acts transitively on X; and

G-foliated space if it is transversely modeled by local left translations in G.

Topological Molino's theory

The first main result of this work is the following topological version of the minimal case in Molino's theory.

Theorem A. *Let $X \equiv (X, \mathcal{F})$ be a compact Polish foliated space, and $\mathcal{H}$ its holonomy pseudogroup. Suppose that X is minimal and equicontinuous, and $\overline{\mathcal{H}}$ is strongly quasi-analytic. Then there is a compact Polish foliated space*

$\widehat{X}_0 \equiv (\widehat{X}_0, \widehat{\mathcal{F}}_0)$*, a foliated map* $\hat{\pi}_0 : \widehat{X}_0 \to X$*, and a local group* G *with a left-invariant metric such that:*

- $\widehat{X}_0$ *is a minimal* G*-foliated space;*
- *the fibers of* $\hat{\pi}_0$ *are homeomorphic to each other; and*
- *the restrictions of* $\hat{\pi}_0$ *to the leaves are the holonomy covers of the leaves of* $\mathcal{F}$*.*

The main difficulty to prove Theorem A is that there is no normal bundle of $\mathcal{F}$, whilst $\widehat{X}_0$ is defined as a subbundle of the bundle of orthonormal frames in the Riemannian foliation case.

To define $\widehat{X}_0$, we first construct what should be its holonomy pseudogroup $\widehat{\mathcal{H}}_0$ on a space $\widehat{T}_0$. To some extent, this was achieved by Álvarez and Candel [5], proving that, with the assumptions of Theorem A, like in the foliation case, there is a local group G, a compact subgroup $K \subset G$ and a dense finitely generated sub-local group $\Gamma \subset G$ such that $\mathcal{H}$ is equivalent to the pseudogroup generated by the local action of Γ on G/K. Hence $\widehat{\mathcal{H}}_0$ should be the pseudogroup generated local action of Γ on G. This may look as a big step towards the proof, but the realization of compactly generated pseudogroups as holonomy pseudogroups of compact foliated spaces is impossible in general, as shown by Meigniez [32]. This difficulty is overcome as follows.

Take a "good" cover of X by distinguished open sets, $\{U_i\}$, with corresponding distinguished submersions $p_i : U_i \to T_i$, and elementary holonomy transformations $h_{ij} : T_{ij} \to T_{ji}$, where $T_{ij} = p_i(U_i \cap U_j)$. Let $\mathcal{H}$ denote the corresponding representative of the holonomy pseudogroup on $T = \bigsqcup_i T_i$, generated by the maps h_{ij}. Then the construction of $\widehat{\mathcal{H}}_0$ must be associated to $\mathcal{H}$ in a natural way, so that it becomes induced by some "good" cover by distinguished open sets of a compact foliated space. In the Riemannian foliation case, the good choices of $\widehat{T}_0$ and $\widehat{\mathcal{H}}_0$ are the following ones:

- Consider an $\mathcal{H}$-invariant metric on T. Fix an orthonormal frame $\hat{x}_0$ at some point x_0 in T. Then $\widehat{T}_0$ is the closure of

  $$\{\, h_*(\hat{x}_0) \mid h \in \mathcal{H},\ x_0 \in \operatorname{dom} h \,\}$$

  in the bundle of orthonormal frames; thus

  $$\widehat{T}_0 = \{\, g_*(\hat{x}_0) \mid g \in \overline{\mathcal{H}},\ x_0 \in \operatorname{dom} g \,\}\ . \tag{$*$}$$

- $\widehat{\mathcal{H}}_0$ is generated by the differentials of the maps in $\mathcal{H}$.

These differential concepts can be modified in the following way:

- In $(*)$, each $g_*(\hat{x}_0)$ determines the germ of g at x_0, $\boldsymbol{\gamma}(g, x_0)$, by the strong quasi-analyticity of $\overline{\mathcal{H}}$. Therefore it also determines $\boldsymbol{\gamma}(f, x)$, where $f = g^{-1}$ and $x = g(x_0)$. So

$$\widehat{T}_0 \equiv \{ \boldsymbol{\gamma}(f, x) \mid f \in \overline{\mathcal{H}},\ x \in \operatorname{dom} f,\ f(x) = x_0 \} . \qquad (**)$$

- The projection $\hat{\pi}_0 : \widehat{T}_0 \to T$ corresponds via $(**)$ to the source map $\boldsymbol{\gamma}(f, x) \mapsto x$.

- The differentials of maps $h \in \mathcal{H}$, acting on orthonormal references, correspond via $(**)$ to the maps $\hat{h}$ defined by

$$\hat{h}(\boldsymbol{\gamma}(f, x)) = \boldsymbol{\gamma}(fh^{-1}, h(x)) .$$

- The topology of $\widehat{T}_0$ can be described via $(**)$ as follows. Let $\overline{S}$ be a pseudo$*$group generating $\overline{\mathcal{H}}$ and satisfying the strong quasi-analyticity condition. Endow $\overline{S}$ with the compact-open topology on partial maps with open domains, as defined by Abd-Allah-Brown [1], and consider the subspace

$$\overline{S} * T = \{ (f, x) \in \overline{S} \mid x \in \operatorname{dom} f\} \subset \overline{S} \times T .$$

  Then the topology of $\widehat{T}_0$ corresponds via $(**)$ to the quotient topology by the germ map $\boldsymbol{\gamma} : \overline{S} * T \to \boldsymbol{\gamma}(\overline{S} * T) \equiv \widehat{T}_0$, which is different from the sheaf topology on germs.

This point of view (replacing orthonormal frames by germs) can be readily translated to the foliated space setting, obtaining a psedogroup $\widehat{\mathcal{H}}_0$ on $\widehat{T}_0$.

Now, consider triples (x, i, γ), where $x \in U_i$, $\gamma \in \widehat{T}_{i,0} := \hat{\pi}_0^{-1}(T_i)$ and $p_i(x) = \hat{\pi}_0(\gamma)$. Declare $(x, i, \gamma) \sim (y, j, \delta)$ if $x = y$ and $\delta = \widehat{h_{ij}}(\gamma)$. Then $\widehat{X}_0$ is defined as the corresponding quotient space; the equivalence class of each triple (x, i, γ) is denoted by $[x, i, \gamma]$. The foliated structure $\widehat{\mathcal{F}}_0$ on $\widehat{X}_0$ is determined by requiring that, for each fixed index i, the elements of the type $[x, i, \gamma]$ form a distinguished open set $\widehat{U}_{i,0}$, with distinguished submersion $\hat{p}_{i,0} : \widehat{U}_{i,0} \to \widehat{T}_{i,0}$ given by $\hat{p}_{i,0}([x, i, \gamma]) = \gamma$. The projection $\hat{\pi}_0 : \widehat{X}_0 \to X$ is defined by $\hat{\pi}_0([x, i, \gamma]) = x$. The properties stated in Theorem A are satisfied with these definitions.

Up to foliated homeomorphisms (respectively, local isomorphisms), $\widehat{X}_0$ (respectively, G) is independent of the choices involved. Hence G can be called the *structural local group* of $\mathcal{F}$.

Growth of equicontinuous foliated spaces

Let us say that a local group G *can be approximated by nilpotent local Lie groups* if, in any identity neighborhood, there exists a sequence of compact normal subgroups F_n such that $F_{n+1} \subset F_n$, $\bigcap_n F_n = \{e\}$ and G/F_n is a nilpotent local Lie group. Our second main result, is the following weaker topological version of the above theorems of Carrière and Breuillard-Gelander.

Theorem B. *Let X be a foliated space satisfying the conditions of Theorem A, and let G be its structural local group. Then one of the following properties holds:*

- *G can be approximated by nilpotent local Lie groups; or*
- *the holonomy covers of all leaves of X have exponential growth.*

Like in the case of Riemannian foliations, Theorem A reduces the proof of Theorem B to the case of G-foliated spaces, where it becomes a problem about local groups. Then, since any local group can be approximated by local Lie groups in the above sense, the result follows by applying the same arguments as Breuillard-Gelander.

Chapter 3
Preliminaries

Contents

3.1 Compact-open topology on partial maps

Most of the contents of this section are taken from [1].

Given spaces X and Y, let $C(X,Y)$ be the space of all continuous maps $X \to Y$; the notation $C_{\text{c-o}}(X,Y)$ may be used to indicate that $C(X,Y)$ is equipped with the compact-open topology. Let Y^* be the space $Y \cup \{\omega\}$, where $\omega \notin Y$, endowed with the topology in which $U \subset Y^*$ is open if and only if $U = Y^*$ or U is open in Y. A *partial map* $X \rightarrowtail Y$ is a continuous map of a subset of X to Y; the set of all partial maps $X \rightarrowtail Y$ is denoted by $\operatorname{Par}(X,Y)$. A partial map $X \rightarrowtail Y$ with open domain is called a *paro map*, and the set of all paro maps $X \rightarrowtail Y$ is denoted by $\operatorname{Paro}(X,Y)$. There is a bijection $\mu : \operatorname{Paro}(X,Y) \to C(X,Y^*)$ defined by

$$\mu(f)(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f \\ \omega & \text{if } x \notin \operatorname{dom} f\ . \end{cases}$$

The topology on $\mathrm{Paro}(X,Y)$ which makes $\mu : \mathrm{Paro}(X,Y) \to C_{\text{c-o}}(X,Y^*)$ a homeomorphism is called the *compact-open topology*, and the notation $\mathrm{Paro}_{\text{c-o}}(X,Y)$ may be used for the corresponding space. This topology has a subbasis of open sets of the form

$$\mathcal{N}(K,O) = \{\, h \in \mathrm{Paro}(X,Y) \mid K \subset \operatorname{dom} h,\ h(K) \subset O \,\}\ ,$$

where $K \subset X$ is compact and $O \subset Y$ is open.

Proposition 3.1.1. *If X is second countable and locally compact, and Y is locally compact, then $\mathrm{Paro}_{\text{c-o}}(X,Y)$ is second countable.*

Proof. By hypothesis, there are countable bases of open sets, $\mathcal{V}$ of X and $\mathcal{W}$ of Y, such that $\overline{V}$ is compact for all $V \in \mathcal{V}$. Then the sets $\mathcal{N}(\overline{V},W)$ ($V \in \mathcal{V}$ and $W \in \mathcal{W}$) form a countable basis of open sets of $\mathrm{Paro}_{\text{c-o}}(X,Y)$. □

The following result is elementary.

Proposition 3.1.2. *For any open $U \subset X$, the restriction of topology of $\mathrm{Paro}_{\text{c-o}}(X,Y)$ to the subset $C(U,Y)$ is its usual compact-open topology.*

Since paro maps are not globally defined, let us make precise the definition of their composition. Given spaces X, Y and Z, the *composition* of two paro maps, $f \in \mathrm{Paro}(X,Y)$ and $g \in \mathrm{Paro}(Y,Z)$, is the paro map $gf \in \mathrm{Paro}(X,Z)$ defined as the usual composition of the maps

$$f^{-1}(\operatorname{dom} g) \xrightarrow{f} \operatorname{dom} g \xrightarrow{g} Z\ .$$

Proposition 3.1.3 (Abd-Allah-Brown [1, Proposition 3])**.** *The following properties hold:*

(*i*) *Let $h : T \rightarrowtail X$ and $g : Y \rightarrowtail Z$ be paro maps. Then the maps*

$$g_* : \mathrm{Paro}_{\text{c-o}}(X,Y) \to \mathrm{Paro}_{\text{c-o}}(X,Z)\ , \quad f \mapsto gf\ ,$$
$$h^* : Paro_{\text{c-o}}(X,Y) \to \mathrm{Paro}_{\text{c-o}}(T,Y)\ , \quad f \mapsto fh\ ,$$

are continuous.

(*ii*) *Let $X' \subset X$ and $Y' \subset Y$ be subspaces such that X' is open in X. Then the map*

$$\mathrm{Paro}_{\text{c-o}}(X',Y') \to \mathrm{Paro}_{\text{c-o}}(X,Y)\ ,$$

mapping a paro map $X' \rightarrowtail Y'$ to the paro map $X \rightarrowtail Y$ with the same graph, is an embedding.

The topology of $\mathrm{Paro}_{\text{c-o}}(X, Y)$ is interesting even when Y is a singleton space; say the space $\mathbf{1} = \{1\}$. Let $\mathcal{O}(X)$ be the family of open subsets of X. There is a natural bijection $\chi : \mathcal{O} \to \mathrm{Paro}(X, \mathbf{1})$ such that $\chi(U)$ is the constant paro map $X \rightarrowtail \mathbf{1}$ with domain U. The topology on $\mathcal{O}(X)$ which makes χ a homeomorphism has a subbasis of open sets of the form

$$W(K) = \{\, U \in \mathcal{O}(X) \mid K \subset U \,\} ,$$

where $K \subset X$ is compact.

Proposition 3.1.4 (Abd-Allah-Brown [1, Proposition 4]). *The domain map* $\mathrm{dom} : \mathrm{Paro}(X, Y) \to \mathcal{O}(X)$, $f \mapsto \mathrm{dom}\, f$, *is continuous.*

Recall that the usual *exponential function*

$$\theta : C_{\text{c-o}}(X \times Y, Z) \to C_{\text{c-o}}(X, C_{\text{c-o}}(Y, Z)) , \quad \theta(f)(x)(y) = f(x, y) ,$$

is a well-defined injection. The pair (X, Y) is called an *exponential pair* if the above θ is surjective for any space Z; it is standard that this happens if Y is locally compact or X is a Hausdorff k-space[1]. The following result is the exponential law for paro maps.

Proposition 3.1.5 (Abd-Allah-Brown [1, Theorem 5]). *The exponential function*

$$\begin{gathered} \theta : \mathrm{Paro}_{\text{c-o}}(X \times Y, Z) \to \mathrm{Paro}_{\text{c-o}}(X, \mathrm{Paro}_{\text{c-o}}(Y, Z)) , \\ \theta(f)(x)(y) = f(x, y) , \end{gathered}$$

is a well defined injection. Further:

(*i*) *if (X, Y) is an exponential pair, then θ is surjective;*

(*ii*) *if X is Hausdorff, then θ is continuous; and,*

(*iii*) *if X and Y are Hausdorff, then θ is an embedding.*

For each $x \in X$, the *x-section* of any $U \subset X \times Y$ is

$$U_x = \{\, y \in Y \mid (x, y) \in U \,\} .$$

If U is open in $X \times Y$, then U_x is open in Y.

Proposition 3.1.6 (Abd-Allah-Brown [1, Proposition 6]). *The section map*

$$\sigma : \mathcal{O}(X \times Y) \to C_{\text{c-o}}(X, \mathcal{O}(Y)) , \quad \sigma(U)(x) = U_x ,$$

is a well-defined injection. Further:

[1]Recall that a *k-space* (or *compactly generated space*) is a topological space where a subset A is closed if and only if $A \cap K$ is closed in K for all compact subspaces K.

(*i*) *if* (X,Y) *is an exponential pair, then* σ *is surjective;*

(*ii*) *if* X *is Hausdorff, then* σ *is continuous; and,*

(*iii*) *if* X *and* Y *are Hausdorff, then* σ *is an embedding.*

Example 3.1.7. The map

$$\mathbb{R} \to \mathrm{Paro}_{\text{c-o}}(\mathbb{R},\mathbb{R})\ , \quad y \mapsto (x \mapsto \log(x+y))\ ,$$

is continuous, as follows from Proposition 3.1.5. The domain of the paro map $x \mapsto \log(x+y)$ is $(-y,\infty)$. So the map

$$\mathbb{R} \to \mathcal{O}(\mathbb{R})\ , \quad y \mapsto (-y,\infty)\ ,$$

is continuous by Proposition 3.1.4; this also follows from Proposition 3.1.6.

Proposition 3.1.8 (Abd-Allah-Brown [1, Proposition 7])**.** *If* Y *is locally compact, then the evaluation partial map*

$$\mathrm{ev} : \mathrm{Paro}_{\text{c-o}}(Y,Z) \times Y \rightarrowtail Z\ , \quad \mathrm{ev}(f,y) = f(y)\ ,$$

is a paro map; in particular, its domain is open.

Proposition 3.1.9 (Abd-Allah-Brown [1, Proposition 8])**.** *If* Y *is locally compact, then the membership relation*

$$M = \{\, (U,y) \in \mathcal{O}(Y) \times Y \mid U \ni y \,\}$$

is open in $\mathcal{O}(Y) \times Y$*.*

Proposition 3.1.10 (Abd-Allah-Brown [1, Proposition 9])**.** *If* X *and* Y *are locally compact, then the composition mapping*

$$\mathrm{Paro}_{\text{c-o}}(X,Y) \times \mathrm{Paro}_{\text{c-o}}(Y,Z) \to \mathrm{Paro}_{\text{c-o}}(X,Y)\ , \quad (f,g) \mapsto gf\ ,$$

is continuous.

Let $\mathrm{Loct}(T)$ be the family of all homeomorphisms between open subsets of a space T, which are called *local transformations*. For $h, h' \in \mathrm{Loct}(T)$, the composition $h'h \in \mathrm{Loct}(T)$ is the composition of maps

$$h^{-1}(\mathrm{im}\, h \cap \mathrm{dom}\, h') \xrightarrow{\ h\ } \mathrm{im}\, h \cap \mathrm{dom}\, h' \xrightarrow{\ h'\ } h'(\mathrm{im}\, h \cap \mathrm{dom}\, h')\ .$$

Each $h \in \mathrm{Loct}(T)$ can be identified with the paro map $T \rightarrowtail T$ with the same graph. This gives rise to a canonical injection $\mathrm{Loct}(T) \to \mathrm{Paro}(T,T)$ compatible with composition. The corresponding restriction of the compact-open topology of $\mathrm{Paro}(T,T)$ to $\mathrm{Loct}(T)$ is also called *compact-open topology*, and the notation $\mathrm{Loct}_{\text{c-o}}(T)$ may be used for the corresponding space. The *bi-compact-open topology* is the smallest topology on $\mathrm{Loct}(X)$ so that the identity and inversion maps

$$\mathrm{Loct}(T) \to \mathrm{Loct}_{\text{c-o}}(T)\ , \quad f \mapsto f^{\pm 1}\ ,$$

are continuous, and the notation $\mathrm{Loct}_{\text{b-c-o}}(T)$ will be used for the corresponding space. The following result is elementary.

Proposition 3.1.11 (Abd-Allah-Brown [1, Proposition 10])**.** *If T is locally compact, then the composition and inversion maps,*

$$\mathrm{Loct}_{\text{b-c-o}}(T) \times \mathrm{Loct}_{\text{b-c-o}}(T) \to \mathrm{Loct}_{\text{b-c-o}}(T)\ , \quad (g, f) \mapsto gf\ ,$$
$$\mathrm{Loct}_{\text{b-c-o}}(T) \to \mathrm{Loct}_{\text{b-c-o}}(T)\ , \quad f \mapsto f^{-1}\ ,$$

are continuous.

3.2 Pseudogroups

Definition 3.2.1 (Sacksteder [41], Haefliger [25])**.** A *pseudogroup* on a space T is a collection $\mathcal{H} \subset \mathrm{Loct}(T)$ such that:

- the identity map of T belongs to $\mathcal{H}$ ($\mathrm{id}_T \in \mathcal{H}$);
- if $h, h' \in \mathcal{H}$, then the composite $h'h$ is in $\mathcal{H}$ ($\mathcal{H}^2 \subset \mathcal{H}$);
- $h \in \mathcal{H}$ implies that $h^{-1} \in \mathcal{H}$ ($\mathcal{H}^{-1} \subset \mathcal{H}$);
- if $h \in \mathcal{H}$ and U is open in $\operatorname{dom} h$, then the restriction $h : U \to h(U)$ is in $\mathcal{H}$; and,
- if a combination (union) of maps in $\mathcal{H}$ is defined and is a homeomorphism, then it is in $\mathcal{H}$.

Remark 1. The following properties hold:

- $\mathrm{id}_U \in \mathcal{H}$ for every open subset $U \subset T$.
- A local transformation $h \in \mathrm{Loct}(T)$ belongs to $\mathcal{H}$ if and only if it locally belongs to $\mathcal{H}$ (any point $x \in \operatorname{dom} h$ has a neighborhood $V_x \subset \operatorname{dom} h$ such that $h|_{V_x} \in \mathcal{H}$).
- Any intersection of pseudogroups on T is a pseudogroup on T.

Example 3.2.2. $\mathrm{Loct}(T)$ is a pseudogroup that contains any other pseudogroup on T. All isometries between open subsets of a Riemannian manifold form a pseudogroup. All symplectic isomorphisms between open subsets of a symplectic manifold form a pseudogroup.

Definition 3.2.3. A *sub-pseudogroup* of a pseudogroup $\mathcal{H}$ on T is a pseudogroup on T contained in $\mathcal{H}$. The *restriction* of $\mathcal{H}$ to an open subset $U \subset T$ is the pseudogroup

$$\mathcal{H}|_U = \{\, h \in \mathcal{H} \mid \operatorname{dom} h \cup \operatorname{im} h \subset U \,\} \ .$$

The pseudogroup *generated* by a set $S \subset \mathrm{Loct}(T)$ is the intersection of all pseudogroups that contain S (the smallest pseudogroup on T containing S).

Definition 3.2.4. Let $\mathcal{H}$ be a pseudogroup on T. The *orbit* of each $x \in T$ is the set

$$\mathcal{H}(x) = \{\, h(x) \mid h \in \mathcal{H},\ x \in \operatorname{dom} h \,\} \ .$$

The orbits form a partition of T. The space of orbits, equipped with the quotient topology, is denoted by $T/\mathcal{H}$. It is said that $\mathcal{H}$ is:

- (*topologically*) *transitive* if some orbit is dense; and
- *minimal* when all of its orbits are dense.

Remark 2. The concept of pseudogroup is a generalization of a group action (the transformations of any group action generate a pseudogroup). Thus pseudogroups can be considered as some kind of generalized dynamical systems. In that sense, all basic dynamical notions have obvious versions for pseudogroups, like the above concepts of orbit, transitiveness and minimality.

The following notion, weaker than the concept of pseudogroup, is useful to study some properties of pseudogroups.

Definition 3.2.5 (Matsumoto [30])**.** A *pseudo∗group* on a space T is a family $S \subset \mathrm{Loct}(T)$ that is closed by the operations of composition and inversion.

Remark 3. Any intersection of pseudo∗groups on T is a pseudo∗group on T.

Definition 3.2.6. A *sub-pseudo∗group* of a pseudo∗group S on T is a pseudo∗group contained in S. The pseudo∗group *generated* by a set $S_0 \subset \mathrm{Loct}(T)$ is the intersection of all pseudo∗groups containing S_0 (the smallest pseudo∗group containing S_0).

Remark 4. Let S be a pseudo∗group on T, and let S_1 be the collection of restrictions of all maps in S to all open subsets of their domains. Then S_1 is also a pseudo∗group on T, and S is a sub-pseudo∗group of S_1.

Definition 3.2.7. In Remark 4, it will be said that S_1 is the *localization* of S. If $S = S_1$, then the pseudo∗group S is called *local*.

Remark 5. Let $S_0 \subset \mathrm{Loct}(T)$. The pseudo∗group S generated by S_0 consists of all compositions of maps in S_0 and their inverses. The pseudo-group $\mathcal{H}$ generated by S_0 consists of all $h \in \mathrm{Loct}(T)$ that locally belong to the localization of S.

Remark 6. If two local pseudo$*$groups, S_1 and S_2, generate the same pseudogroup $\mathcal{H}$, then $S_1 \cap S_2$ is also a local pseudo$*$group that generates $\mathcal{H}$

Let $\mathcal{H}$ and $\mathcal{H}'$ be pseudogroups on respective spaces T and T'.

Definition 3.2.8 (Haefliger [22, 23])**.** A *morphism*[2] $\Phi\colon \mathcal{H} \to \mathcal{H}'$ is a maximal collection Φ of homeomorphisms of open sets of T to open sets of T' such that:

- if $\varphi \in \Phi$, $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$, then $h'\varphi h \in \Phi$ ($\mathcal{H}'\Phi\mathcal{H} \subset \Phi$);
- the family of the domains of maps in Φ form an open covering of T; and
- if $\varphi, \varphi' \in \Phi$, then $\varphi'\varphi^{-1} \in \mathcal{H}'$ ($\Phi\Phi^{-1} \subset \mathcal{H}'$).

A morphism Φ is called an *equivalence* if the family $\Phi^{-1} = \{\, \varphi^{-1} \mid \varphi \in \Phi \,\}$ is also a morphism.

Remark 7. An equivalence $\Phi : \mathcal{H} \to \mathcal{H}'$ can be characterized as a maximal family of homeomorphisms of open sets of T to open sets of T' such that:

- $\mathcal{H}'\Phi\mathcal{H} \subset \Phi$;
- $\Phi\Phi^{-1}$ generates $\mathcal{H}'$; and
- $\Phi^{-1}\Phi$ generates $\mathcal{H}$.

Remark 8. Any morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ induces a map between the corresponding orbit spaces, $T/\mathcal{H} \to T/\mathcal{H}'$. This map is a homeomorphism if Φ is an equivalence.

Definition 3.2.9. Let Φ_0 be a family of homeomorphisms of open subsets of T to open subsets of T' such that:

- the union of domains of maps in Φ_0 meet all $\mathcal{H}$-orbits; and
- $\Phi_0\mathcal{H}\Phi_0^{-1} \subset \mathcal{H}'$.

Then there is a unique morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ containing Φ_0, which is said to be *generated* by Φ_0. If moreover:

- the union of images of maps in Φ_0 meet all $\mathcal{H}'$-orbits; and
- $\Phi_0^{-1}\mathcal{H}\Phi_0 \subset \mathcal{H}$;

then Φ is an equivalence.

Definition 3.2.10 (Haefliger [25])**.** A pseudogroup $\mathcal{H}$ on a locally compact space T is said to be *compactly generated* if:

[2]This is usually called *étalé morphism*. We simply call it morphism because no other type of morphism will be considered here.

- there is a relatively compact open subset $U \subset T$ meeting each $\mathcal{H}$-orbit;
- there is a finite set $S = \{h_1, \dots, h_n\} \subset \mathcal{H}|_U$ that generates $\mathcal{H}|_U$; and
- each h_i is the restriction of some $\tilde{h}_i \in \mathcal{H}$ with $\overline{\operatorname{dom} h_i} \subset \operatorname{dom} \tilde{h}_i$.

Remark 9. Compact generation is very delicate to study (see [31]).

Definition 3.2.11 (Haefliger [22]). A pseudogroup $\mathcal{H}$ is called *quasi-analytic* if every $h \in \mathcal{H}$ is the identity around some $x \in \operatorname{dom} h$ whenever h is the identity on some open set whose closure contains x.

Example 3.2.12. The pseudogroup of local analytic transformations of an analytic manifold is quasi-analytic. The pseudogroup of local isometries of a Riemannian manifold is quasi-analytic because every local isometry with connected domain is determined by its differential at any given point.

If a pseudogroup $\mathcal{H}$ on a space T is quasi-analytic, then every $h \in \mathcal{H}$ with connected domain is the identity on $\operatorname{dom} h$ if it is the identity on some non-empty open set. Because of this, quasi-analyticity is interesting when T is locally connected, but local connectivity is too restrictive in our setting. Then, instead of requiring local connectivity, the following stronger version of quasi-analyticity will be used.

Definition 3.2.13 (Álvarez-Candel [4]). A pseudogroup $\mathcal{H}$ on a space T is said to be *strongly quasi-analytic* if it is generated by some sub-pseudo$*$group $S \subset \mathcal{H}$ such that any transformation in S is the identity on its domain if it is the identity on some non-empty open subset of its domain.

Remark 10. In [4], the term used for the above property is "quasi-effective". However the term "strongly quasi-analytic" seems to be more appropriate.

Remark 11. If the condition on $\mathcal{H}$ to be strongly quasi-analytic is satisfied with a sub-pseudo$*$group S, it is also satisfied with the localization of S.

Definition 3.2.14 (Haefliger [22]). A pseudogroup $\mathcal{H}$ on a space T is said to be *complete* if, for all $x, y \in T$, there are compact open neighborhoods, U_x of x and V_y of y, such that, for all $h \in \mathcal{H}$ and $z \in U_x \cap \operatorname{dom} h$ with $h(z) \in V_y$, there is some $g \in \mathcal{H}$ such that $\operatorname{dom} g = U_x$ and $\boldsymbol{\gamma}(g, z) = \boldsymbol{\gamma}(h, z)$.

Example 3.2.15. Any pseudogroup induced by a group action is obviously complete.

Example 3.2.16 (Haefliger [25]). Any compactly generated pseudogroup of local isometries on a Riemannian manifold is complete.

Remark 12. Completeness is not invariant by pseudogroup equivalences. For instance, the pseudogroup $\mathcal{H}$ on $\mathbb{R}$ generated by any homothety $x \mapsto \lambda x$ ($\lambda < 1$) is complete, and the open subset $U = (-1, 1) \subset \mathbb{R}$ cuts all of its orbits, however $\mathcal{H}|_U$ is not complete.

Since any pseudo$*$group S on T is a sub-pseudo$*$group of $\mathrm{Loct}(T)$, it can be endowed with the restriction of the (bi-)compact-open topology, also called (*bi-*)*compact-open topology* of S, and the notation $S_{\text{(b-)c-o}}$ may be used for the corresponding space. In this way, according to Proposition 3.1.11, if T is locally compact, then $S_{\text{b-c-o}}$ becomes a *topological pseudo$*$group* in the sense that the composition and inversion maps of S are continuous. In particular, this applies to a pseudogroup $\mathcal{H}$ on T, obtaining $\mathcal{H}_{\text{(b-)c-o}}$. In particular, $\mathcal{H}_{\text{b-c-o}}$ is a *topological pseudogroup* in the same sense as above if T is locally compact.

Remark 13. If S is a sub-pseudo$*$group of S', then $S_{\text{(b-)c-o}} \hookrightarrow S'_{\text{(b-)c-o}}$ is continuous.

Recall that a topological space is called *Polish* if it is separable and completely metrizable. This condition is sometimes assumed to get better dynamical properties. Thus the pseudogroups considered from now on will be assumed to act on locally compact Polish spaces; these spaces can be characterized by the condition of being locally compact, Hausdorff and second countable [29, Theorem 5.3].

3.3 Groupoid of germs of a pseudogroup

Definition 3.3.1. A *groupoid* $\mathfrak{G}$ is a small category where every morphism is an isomorphism. This means that $\mathfrak{G}$ is a set (of *morphisms*) equipped with the structure defined by an additional set T (of *objects*), and the following *structural* maps:

- the *source* and *target* maps $s, t : \mathfrak{G} \to T$;
- the *unit* map $T \to \mathfrak{G}$, $x \mapsto 1_x$;
- the *operation* (or *multiplication*) map $\mathfrak{G} \times_T \mathfrak{G} \to \mathfrak{G}$, $(\delta, \gamma) \mapsto \delta\gamma$, where
$$\mathfrak{G} \times_T \mathfrak{G} = \{\, (\delta, \gamma) \in \mathfrak{G} \times \mathfrak{G} \mid t(\gamma) = s(\delta) \,\} \subset \mathfrak{G} \times \mathfrak{G} \;;$$
- and the *inversion* map $\mathfrak{G} \to \mathfrak{G}$, $\gamma \mapsto \gamma^{-1}$;

such that the following conditions are satisfied:

- $s(\delta\gamma) = s(\gamma)$ and $t(\delta\gamma) = t(\delta)$ for all $(\delta, \gamma) \in \mathfrak{G} \times_T \mathfrak{G}$.
- For all $\gamma, \delta, \varepsilon \in \mathfrak{G}$ with $t(\gamma) = s(\delta)$ and $t(\delta) = s(\varepsilon)$, we have $\varepsilon(\delta\gamma) = (\varepsilon\delta)\gamma$ (associativity).

- $1_{t(\gamma)}\gamma = \gamma 1_{s(\gamma)} = \gamma$ (units or identity elements).
- $s(\gamma) = t(\gamma^{-1})$, $t(\gamma) = s(\gamma^{-1})$, $\gamma^{-1}\gamma = 1_{s(\gamma)}$ and $\gamma\gamma^{-1} = 1_{t(\gamma)}$ for all $\gamma \in \mathfrak{G}$ (inverse elements).

If moreover $\mathfrak{G}$ and T are equipped with topologies so that all of the above structural maps are continuous, then $\mathfrak{G}$ is called a *topological groupoid*.

Remark 14. For a groupoid $\mathfrak{G}$, observe that $s(1_x) = t(1_x) = x$ for all $x \in T$, and therefore the source and target maps $s, t : \mathfrak{G} \to T$ are surjective, and the unit map $T \to \mathfrak{G}$ is injective. If moreover $\mathfrak{G}$ is a topological groupoid, then the unit map $T \to \mathfrak{G}$ is a topological embedding, and therefore the topology of T is determined by the topology of $\mathfrak{G}$; indeed, we can consider T as a subspace of $\mathfrak{G}$ if desired.

Definition 3.3.2. A topological grupoid is called *étalé* if the source and target maps are local homeomorphisms.

Example 3.3.3. Let X be a locally compact space. The *homotopy groupoid* $\Pi(X)$ of X is the quotient space of the path space $C_{\text{c-o}}([0,1], X)$ by the relation of homotopy relative to the end points. Its groupoid structure is given by the multiplication and inversion induced by those operations on paths, its unit space is X, the unit injection $X \to \Pi(X)$ is defined by the constant paths, and the source and target mas $\Pi(X) \to X$ are given by taking the origin and final point of each path.

Example 3.3.4. A continuous left action of a topological group G on a space X induces a topological groupoid structure on $G \times X$, equipped with the product topology, with the unit space X, the unit injection $X \to G \times X$ given by $x \mapsto (1, x)$, the source and target maps $s, t : G \times X \to X$ defined by $s(g, x) = x$ and $t(g, x) = g \cdot x$, the operation given by $(h, y)(g, x) = (hg, x)$ if $y = g \cdot x$, and the inversion defined by $(g, x)^{-1} = (g^{-1}, g \cdot x)$.

Let $\mathcal{H}$ be a pseudogroup on a space T. For $h, h' \in \mathcal{H}$ and $x \in \operatorname{dom} h \cap \operatorname{dom} h'$, write $(h, x) \sim (h', x)$ if there is a neighborhood U of x in $\operatorname{dom} h \cap \operatorname{dom} h'$ such that $h|_U = h'|_U$. This defines an equivalence relation on the set

$$\mathcal{H} * T = \{\, (h, x) \in \mathcal{H} \times T \mid x \in \operatorname{dom} h \,\} \subset \mathcal{H} \times T \,.$$

Note that $\mathcal{H} * T$ is the domain of the evaluation partial map $\mathrm{ev} : \mathcal{H} \times T \rightarrowtail T$. The equivalence class of each $(h, x) \in \mathcal{H} * T$ is called the *germ* of h at x, which will be denoted by $\boldsymbol{\gamma}(h, x)$. The corresponding quotient set is denoted by $\mathfrak{G}$, and the quotient map, $\boldsymbol{\gamma} : \mathcal{H} * T \to \mathfrak{G}$, is called the *germ map*. It is well known that $\mathfrak{G}$ is a groupoid with set of units T, where the source and target maps $s, t : \mathfrak{G} \to T$ are given by $s(\boldsymbol{\gamma}(h, x)) = x$ and $t(\boldsymbol{\gamma}(h, x)) = h(x)$, the unit map $T \to \mathfrak{G}$ is defined by $1_x = \boldsymbol{\gamma}(\mathrm{id}_T, x)$, the operation map $\mathfrak{G} \times_T \mathfrak{G} \to \mathfrak{G}$ given by

$$\boldsymbol{\gamma}(g, h(x))\ \boldsymbol{\gamma}(h, x) = \boldsymbol{\gamma}(gh, x)\ ,$$

and the inversion map is defined by

$$\boldsymbol{\gamma}(h, x)^{-1} = \boldsymbol{\gamma}(h^{-1}, h(x))\ ;$$

thus the operation and inversion of $\mathfrak{G}$ are induced by the composition and inversion of maps in $\mathcal{H}$.

For $x, y \in T$, let us use the notation $\mathfrak{G}_x = s^{-1}(x)$, $\mathfrak{G}^y = t^{-1}(y)$ and $\mathfrak{G}_x^y = \mathfrak{G}_x \cap \mathfrak{G}^y$; in particular, the group $\mathfrak{G}_x^x$ will be called the *germ group* of $\mathcal{H}$ at x. Points in the same $\mathcal{H}$-orbit have isomorphic germ groups (if $y \in \mathcal{H}(x)$, an isomorphism $\mathfrak{G}_y^y \to \mathfrak{G}_x^x$ is given by conjugation with any element in $\mathfrak{G}_x^y$); hence the germ groups of the orbits make sense up to isomorphism. Under pseudogroup equivalences, corresponding orbits have isomorphic germ groups. The set $\mathfrak{G}_x$ will be called the *germ cover* of the orbit $\mathcal{H}(x)$ with base point x. The target map restricts to a surjective map $\mathfrak{G}_x \to \mathcal{H}(x)$ whose fibers are bijective to $\mathfrak{G}_x^x$ (if $y \in \mathcal{H}(x)$, a bijection $\mathfrak{G}_x^x \to \mathfrak{G}_x^y$ is given by left product with any element in $\mathfrak{G}_x^y$); thus $\mathfrak{G}_x$ is finite if and only if both $\mathfrak{G}_x^x$ and $\mathcal{H}(x)$ are finite. Moreover germ covers based on points in the same orbit are also bijective (if $y \in \mathcal{H}(x)$, a bijection $\mathfrak{G}_y \to \mathfrak{G}_x$ is given by right product with any element in $\mathfrak{G}_x^y$); therefore the germ covers of the orbits make sense up to bijections.

Definition 3.3.5. It is said that $\mathcal{H}$ is:

- *locally free* if all of its germ groups are trivial (for all $h \in \mathcal{H}$ and $x \in \operatorname{dom} h$ such that $h(x) = x$, we have $\boldsymbol{\gamma}(h, x) = \boldsymbol{\gamma}(\mathrm{id}_T, x)$); and
- *strongly locally free* if $\mathcal{H}$ is generated by a sub-pseudo$*$group $S \subset \mathcal{H}$ such that, for all $h \in S$ and $x \in \operatorname{dom} h$, if $h(x) = x$ then $h = \mathrm{id}_{\operatorname{dom} h}$

Remark 15. The condition of being (strongly) locally free is tronger than the condition of being (strongly) quasi-analytic. If $\mathcal{H}$ is locally free and satisfies the condition of strong quasi-analiticity with a sub-pseudo$*$group $S \subset \mathcal{H}$, generating $\mathcal{H}$, then $\mathcal{H}$ also satisfies the condition of being strongly locally free with S.

Remark 16. If the condition on $\mathcal{H}$ to be strongly locally free is satisfied with a sub-pseudo$*$group S, then it is also satisfied with the localization of S.

The best-known topology on $\mathfrak{G}$ is the *sheaf topology*, but we will not use it. It has a basis given by the sets

$$\{\, \boldsymbol{\gamma}(h, x) \mid x \in \operatorname{dom} h \,\}$$

for $h \in \mathcal{H}$. Equipped with the sheaf topology, $\mathfrak{G}$ becomes an étalé grupoid.

Another topology on $\mathfrak{G}$ can be defined as follows. The set $\mathcal{H} * T$ is open in $\mathcal{H}_{\text{(b-)c-o}} \times T$ by Proposition 3.1.8. It will be denoted by $\mathcal{H}_{\text{(b-)c-o}} * T$ when endowed with the restriction of the topology of $\mathcal{H}_{\text{(b-)c-o}} \times T$. The induced quotient topology on $\mathfrak{G}$, via the germ map $\boldsymbol{\gamma} : \mathcal{H}_{\text{(b-)c-o}} * T \to \mathfrak{G}$, will be also called the *(bi-)compact-open topology*. The corresponding space will be denoted by $\mathfrak{G}_{\text{(b-)c-o}}$, or by $\mathfrak{G}_{\mathcal{H},\text{(b-)c-o}}$ if reference to $\mathcal{H}$ is needed. It follows from Proposition 3.1.11 that $\mathfrak{G}_{\text{b-c-o}}$ is a topological groupoid if T is locally compact.

Suppose that $\mathcal{H}$ is generated by some sub-pseudo$*$group $S \subset \mathcal{H}$. By using S instead of $\mathcal{H}$ as above, we get an open subspace $S_{\text{(b-)c-o}} * T \subset S_{\text{(b-)c-o}} \times T$, which is subspace of $\mathcal{H}_{\text{(b-)c-o}} * T$, so that $\boldsymbol{\gamma} : S_{\text{(b-)c-o}} * T \to \mathfrak{G}$ is surjective. By using this map, we get another quotient topology on $\mathfrak{G}$; it will be also called *(bi-)compact-open topology*. The corresponding space will be denoted by $\mathfrak{G}_{\text{(b-)c-o}}$ as before, or by $\mathfrak{G}_{S,\text{(b-)c-o}}$ if reference to S is needed. As above, $\mathfrak{G}_{\text{b-c-o}}$ is a topological groupoid if T is locally compact. There is a commutative diagram

$$\begin{array}{ccc}
S_{\text{(b-)c-o}} * T & \xrightarrow{\text{inclusion}} & \mathcal{H}_{\text{(b-)c-o}} * T \\
\gamma\downarrow & & \downarrow\gamma \\
\mathfrak{G}_{S,\text{(b-)c-o}} & \xrightarrow{\text{identity}} & \mathfrak{G}_{\mathcal{H},\text{(b-)c-o}}
\end{array}$$

where the top map is an embedding and the vertical maps are identifications. Hence the identity map $\mathfrak{G}_{S,\text{(b-)c-o}} \to \mathfrak{G}_{\mathcal{H},\text{(b-)c-o}}$ is continuous. Similarly, the identity map $\mathfrak{G}_{S,\text{b-c-o}} \to \mathfrak{G}_{S,\text{c-o}}$ is continuous.

Question 3.3.6. When is the identity map $\mathfrak{G}_{S,\text{(b-)c-o}} \to \mathfrak{G}_{\mathcal{H},\text{(b-)c-o}}$ a homeomorphism?

Question 3.3.7. When is the identity map $\mathfrak{G}_{S,\text{b-c-o}} \to \mathfrak{G}_{S,\text{c-o}}$ a homeomorphism?

3.4 Local groups and local actions

Let us recall some notions from [27].

Definition 3.4.1 (See e.g. [27])**.** A local group is a quintuple $G \equiv (G, e, \cdot, ', \mathfrak{D})$ satisfying the following conditions:

(1) $(G, \mathfrak{D})$ is a topological space;

(2) $\cdot$ is a function from a subset of $G \times G$ to G;

(3) $'$ is a function from a subset of G to G;

(4) there is a subset O of G such that

  (a) O is an open neighborhood of e in G,

  (b) $O \times O$ is a subset of the domain of $\cdot$,

  (c) O is a subset of the domain of $'$,

  (d) for all $a, b, c \in O$, if $a \cdot b, b \cdot c \in O$, then $(a \cdot b) \cdot c = (a \cdot b) \cdot c$,

  (e) for all $a \in O$, $a' \in O$, $a \cdot e = e \cdot a = a$ and $a' \cdot a = a \cdot a' = e$,

  (f) the map $\cdot : O \times O \to G$ is continuous,

  (g) the map $' : O \to G$ is continuous;

(5) the set $\{e\}$ is closed in G.

It is a usual convention that asserting that a local group satisfies some topological property means that the property is satisfied on some open neighborhood of e.

A *local homomorphism* of a local group G to a local group H is a continuous partial map $\phi : G \rightarrowtail H$, whose domain is an identity neighborhood in G, which is compatible in the usual sense with the identity elements, the operations and inversions of G and H. If moreover ϕ restricts to a homeomorphism between some identity neighborhoods in G and H, then it is called a *local isomorphism*, and G and H are said to be *locally isomorphic*.

The collection of all sets O satisfying condition (4) will be denoted by ΨG. This is a neighborhood basis of e in G; all of these neighborhoods are symmetric with respect to the inverse operation (3). Let $\Phi(G, n)$ denote the collection of subsets A of G such that the product of any collection of at most n elements of A is defined, and the set A^n of such products is contained in some $O \in \Psi G$.

If G is a local group, then H is a subgroup of G if $H \in \Phi(G, 2)$, $e \in H$, $H' = H$ and $H^2 = H$.

If G is a local group, then $H \subset G$ is a sub-local group of G in case H is itself a local group with respect to the induced operations and topology.

If G is a local group, then ΥG denotes the set of all pairs (H, U) of subsets of G so that:

(1) $e \in H$;

(2) $U \in \Psi G$;

(3) for all $a, b \in U \cap H$, $a \cdot b \in H$; and

(4) $c' \in H$ for all $c \in U \cap H$.

Jacoby [27, Theorem 26] proves that $H \subset G$ is a sub-local group if and only if there exists U such that $(H, U) \in \Upsilon G$.

Let G be a local group and let ΠG denote the family of pairs (H, U) so that:

(1) $e \in H$;

(2) $U \in \Psi G \cap \Phi(G, 6)$;

(3) for all $a, b \in U^6 \cap H$, $a \cdot b \in H$;

(4) for all $c \in U^6 \cap H$, $c' \in H$;

(5) $U^2 \setminus H$ is open.

Given such a pair $(H, U) \in \Pi G$, there is a (completely regular, Hausdorff) space $G/(U, H)$ and a continuous open surjection $T : U^2 \to G/(U, H)$ such that $T(a) = T(b)$ if and only if $a' \cdot b \in H$ (cf. [27, Theorem 29]).

If (H, V) is another pair in ΠG, then the spaces $G/(H, U)$ and $G/(H, V)$ are locally homeomorphic in an obvious way. Thus the concept of coset space of H is well defined in this sense, as a germ of a topological space. The notation G/H will be used in this sense; and to say that G/H has certain topological property will mean that some $G/(H, U)$ has such property.

Let ΔG be the set of pairs (H, U) such that $(H, U) \in \Pi G$ and, for all $a \in H \cap U^4$ and $b \in U^2$, $b' \cdot (a \cdot b) \in H$. A subset $H \subset G$ is called a normal sub-local group of G if there exists U such that $(H, U) \in \Delta G$. If $(H, U) \in \Delta G$ then the quotient space $G/(H, U)$ admits the structure of a local group (see [27, Theorem 35] for the pertinent details) and the natural projection $T : U^2 \to G/(H, U)$ is a local homomorphism. As before, another such pair (H, V) produces a locally isomorphic quotient local group, and the simpler notation G/H may be used.

As usual, $a \cdot b$ and a' will be denoted by ab and a^{-1}.

A local version of Hilbert's 5th problem, asking whether any locally Euclidean local group is a local Lie group, was studied by Jacoby [27]. Below, we state the main theorems of Jacoby [27] leading to its affirmative solution (it is a direct consequence of Theorem 3.4.4). However, as pointed out by Plaut in [39], Jacoby failed to recognize the following subtlety: in local groups, "local associativity" (for three elements) does not imply "global associativity" (for any finite sequence of elements). In fact, Olver [36] gave examples of connected local Lie groups that are not globally associative. Thus the proof of Jacoby is incorrect. Fortunately, a completely new proof of the local Hilbert's 5th problem has been given by Goldbring [19].

Theorem 3.4.2 (Jacoby [27, Theorem 96]; correction by Goldbring [19]). *Any locally compact local group without small subgroups is a local Lie group.*

In the above result, a local group without small subgroups is a local group where some neighborhood of the identity element contains no nontrivial subgroup.

Theorem 3.4.3 (Jacoby [27, Theorems 97–103]; correction by Goldbring [19]). *Any locally compact second countable local group G can be approximated by local Lie groups. More precisely, given $V \in \Psi G \cap \Phi(G, 2)$, there exists $U \in \Psi G$ with $U \subset V$ and there exists a sequence of compact normal subgroups $F_n \subset U$ such that*

(i) $F_{n+1} \subset F_n$,

(ii) $\bigcap_n F_n = \{e\}$,

(iii) $(F_n, U) \in \Delta G$, *and*

(iv) $G/(F_n, U)$ *is a local Lie group.*

Theorem 3.4.4 (Jacoby [27, Theorem 107]; correction by Goldbring [19]). *Any finite dimensional metrizable locally compact local group is locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group.*

All local groups appearing in this paper will be assumed, or proved, to be locally compact and second countable.

Definition 3.4.5. A local group G is a *local transformation group* on a subspace $X \subset Y$ if there is given a continuous map $G \times X \to Y$, written $(g, x) \mapsto gx$, such that

- $ex = x$ for all $x \in X$; and
- $g_1(g_2x) = (g_1g_2)x$, provided both sides are defined.

This map $G \times X \to Y$ is called a *local action* of G on $X \subset Y$.

Remark 17. The local transformations of any local action of a local group on a space generate a pseudogroup.

Example 3.4.6. The typical example of local action is the following. Let H be a sub-local group of G. If $(H, U) \in \Pi G$ and $T : U^2 \to G/(H, U)$ is the natural projection, then U is a sub-local group of G and the map $(u, T(g)) \mapsto T(u \cdot g)$ defines a local action of U on the open subspace $T(U)$ of $G/(H, U)$.

Remark 18. If G is a local group locally acting on $X \subset Y$ and the local action is locally transitive at $x \in X$ in the sense that there is a neighborhood $V \in \Psi G$ such that Vx includes a neighborhood of x in X, then there is a sub-local group H of G and an open subset $U \subset G$ such that $(H, U) \in \Pi G$ and the orbit map $g \in G \mapsto gx \in X$ induces a local homeomorphism $G/(H, U) \to X$ at x, which is equivariant with respect to the action of U.

Theorem 3.4.7 (Ávarez-Candel [4]; this is a local version of [35, Theorem 6.2.2]). *Let G be a locally compact, separable and metrizable local group. Suppose that there is a local action of G on a finite dimensional subspace $X \subset Y$ and that the action is locally transitive at some $x \in X$. Fix some $(H, U) \in \Pi G$ so that the orbit map $g \mapsto gx$ induces a local homeomorphism $G/(H, U) \to X$ at x. Then there exists a connected normal subgroup K of G such that $K \subset H$, $(K, U) \in \Pi G$ and $G/(K, U)$ is finite dimensional.*

3.5 Equicontinuous pseudogroups

Álvarez and Candel introduced the following structure to define equicontinuity for pseudogroups [4]. Let[3] $\{T_i, d_i\}$ be a family of metric spaces such that $\{T_i\}$ is a covering of a set T, each intersection $T_i \cap T_j$ is open in (T_i, d_i) and (T_j, d_j), and, for all $\varepsilon > 0$, there is some $\delta(\varepsilon) > 0$ so that the following property holds: for all i, j and $z \in T_i \cap T_j$, there is some open neighborhood $U_{i,j,z}$ of z in $T_i \cap T_j$ (with respect to the topology induced by d_i and d_j) such that

$$d_i(x, y) < \delta(\varepsilon) \Longrightarrow d_j(x, y) < \varepsilon$$

for all $\varepsilon > 0$ and all $x, y \in U_{i,j,z}$. Such a family is called a *cover of T by quasi-locally equal metric spaces*. Two such families are called *quasi-locally equal* when their union is also a cover of T by quasi-locally equal metric spaces. This is an equivalence relation whose equivalence classes are called *quasi-local metrics* on T. For each quasi-local metric $\mathfrak{Q}$ on T, the pair $(T, \mathfrak{Q})$ is called a *quasi-local metric space*. Such a $\mathfrak{Q}$ induces a topology on T so that, for each $\{T_i, d_i\}_{i \in I} \in \mathfrak{Q}$, the family of open balls of all metric spaces (T_i, d_i) form a basis of open sets. Any topological concept or property of $(T, \mathfrak{Q})$ refers to this underlying topology. A quasi-local metric space $(T, \mathfrak{Q})$ is a locally compact Polish space if and only if it is Hausdorff, paracompact and separable [4].

Definition 3.5.1 (Álvarez-Candel [4]). Let $\mathcal{H}$ be a pseudogroup on a quasi-local metric space $(T, \mathfrak{Q})$. Then $\mathcal{H}$ is said to be *equicontinuous* if there exists some $\{T_i, d_i\}_{i \in I} \in \mathfrak{Q}$ and some sub-pseudo$*$group $S \subset \mathcal{H}$, generating $\mathcal{H}$, such that, for every $\varepsilon > 0$, there is some $\delta(\varepsilon) > 0$ so that

$$d_i(x, y) < \delta(\varepsilon) \Longrightarrow d_j(h(x), h(y)) < \varepsilon$$

[3] The notation will be simplified by using, for instance, $\{T_i, d_i\}$ instead of $\{(T_i, d_i)\}$.

for all $h \in S$, $i, j \in I$ and $x, y \in T_i \cap h^{-1}(T_j \cap \operatorname{im} h)$.

Remark 19. The original term of [4] is "strongly equicontinuous". We use here the simpler term "equicontinuous" because the weak equicontinuity of [4] is not considered here.

Remark 20. If the condition on $\mathcal{H}$ to be equicontinuous is satisfied with a sub-pseudo$*$group S, then it is also satisfied with the localization of S.

Lemma 3.5.2 (Álvarez-Candel [4, Lemma 8.8]). *Let $\mathcal{H}$ and $\mathcal{H}'$ be equivalent pseudogroups on locally compact Polish spaces. Then $\mathcal{H}$ is equicontinuous if and only if $\mathcal{H}'$ is equicontinuous.*

Proposition 3.5.3 (Alvarez-Candel [4, Proposition 8.9]). *Let $\mathcal{H}$ be a compactly generated and equicontinuous pseudogroup on a locally compact Polish quasi-local metric space $(T, \mathfrak{Q})$, and let U be any relatively compact open subset of $(T, \mathfrak{Q})$ that meets every $\mathcal{H}$-orbit. Suppose that $\{T_i, d_i\}_{i\in I} \in \mathfrak{Q}$ satisfies the condition of equicontinuity. Let E be any system of compact generation of $\mathcal{H}$ on U, and let $\bar{g}$ be an extension of each $g \in E$ with $\overline{\operatorname{dom} g} \subset \operatorname{dom} \bar{g}$. Also, let $\{T_i'\}_{i\in I}$ be any shrinking*[4] *of $\{T_i\}_{i\in I}$. Then there is a finite family $\mathcal{V}$ of open subsets of $(T, \mathfrak{Q})$ whose union contains U and such that, for any $V \in \mathcal{V}$, $x \in U \cap V$, and $h \in \mathcal{H}$ with $x \in \operatorname{dom} h$ and $h(x) \in U$, the domain of $\tilde{h} = \bar{g}_n \cdots \bar{g}_1$ contains V for any composite $h = g_n \cdots g_1$ defined around x with $g_1, \dots, g_n \in E$, and moreover $V \subset T_{i_0}'$ and $\tilde{h}(V) \subset T_{i_1}'$ for some $i_0, i_1 \in I$.*

Remark 21. The statement of Proposition 3.5.3 is stronger than the completeness of $\mathcal{H}|_U$. Since we can choose U large enough to contain two arbitrarily given points of T, it follows $\mathcal{H}$ is complete.

Proposition 3.5.4 (Álvarez-Candel [4, Proposition 9.9]). *Let $\mathcal{H}$ be a compactly generated, equicontinuous and strongly quasi-analytic pseudogroup on a locally compact Polish space T. Suppose that the conditions of equicontinuity and strong quasi-analyticity are satisfied with a sub-pseudo$*$group $S \subset \mathcal{H}$, generating $\mathcal{H}$. Let A, B be open subsets of T such that $\overline{A}$ is compact and contained in B. If x and y are close enough points in T, then*

$$f(x) \in A \Rightarrow f(y) \in B$$

for all $f \in S$ whose domain contains x and y.

Theorem 3.5.5 (Álvarez-Candel [4, Theorem 11.11]). *Let $\mathcal{H}$ be a compactly generated and equicontinuous pseudogroup on a locally compact Polish space T. If $\mathcal{H}$ is transitive, then $\mathcal{H}$ is minimal.*

[4]Recall that a *shrinking* of an open cover $\{U_i\}$ of a space X is an open cover $\{U_i'\}$ of X, with the same index set, such that $\overline{U_i'} \subset U_i$ for all i. On the other hand, if $\{U_i\}$ is a cover of a subset $A \subset X$ by open subsets of X, a *shrinking* of $\{U_i\}$, as cover of A by open subsets of X, is a cover $\{U_i'\}$ of A by open subsets of X, with the same index set, such that $\overline{U_i'} \subset U_i$ for all i.

Theorem 3.5.6 (Álvarez-Candel [4, Theorem 12.1]). *Let $\mathcal{H}$ be a strongly quasi-analytic, compactly generated and equicontinuous pseudogroup on a locally compact Polish space T. Let $S \subset \mathcal{H}$ be a sub-pseudo∗group generating $\mathcal{H}$ and satisfying the conditions of equicontinuity and strong quasi-analyticity. Let $\widetilde{\mathcal{H}}$ be the set of maps h between open subsets of T that satisfy the following property: for every $x \in \operatorname{dom} h$, there exists a neighborhood O_x of x in $\operatorname{dom} h$ so that the restriction $h|_{O_x}$ is in the closure of $C(O_x, T) \cap S$ in $C_{\text{c-o}}(O_x, T)$. Then:*

(*i*) *$\widetilde{\mathcal{H}}$ is closed under composition, combination and restriction to open sets;*

(*ii*) *every map in $\widetilde{\mathcal{H}}$ is a homeomorphism around every point of its domain;*

(*iii*) *$\overline{\mathcal{H}} = \widetilde{\mathcal{H}} \cap \operatorname{Loct}(T)$ is a pseudogroup $\overline{\mathcal{H}}$ that contains $\mathcal{H}$;*

(*iv*) *$\overline{\mathcal{H}}$ is equicontinuous;*

(*v*) *the orbits of $\overline{\mathcal{H}}$ are equal to the closures of the orbits of $\mathcal{H}$; and*

(*vi*) *$\widetilde{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are independent of the choice of S.*

Remark 22. In Theorem 3.5.6, let $\overline{S}$ be the set of local transformations that are in the union of the closures of $C(O, T) \cap S$ in $C_{\text{c-o}}(O, T)$ with O running on the open sets of T. According to the proof of [4, Theorem 12.1], $\overline{S}$ is a pseudo∗group that generates $\overline{\mathcal{H}}$. Moreover, if $\mathcal{H}$ satisfies the equicontinuity condition with S and some representative $\{T_i, d_i\}$ of a quasi-local metric, then $\overline{\mathcal{H}}$ satisfies the equicontinuity condition with $\overline{S}$ and $\{T_i, d_i\}$.

Remark 23. From the proof of [4, Theorem 12.1], it also follows easily that the pseudo∗group $\overline{S}$, defined in Remark 22, satisfies the following property. Any $x \in \overline{U}$ has a neighborhood O in T such that the closure of

$$\{\, h \in C(O, T) \cap S \mid h(O) \cap \overline{U} \neq \emptyset \,\}$$

in $C_{\text{c-o}}(O, T)$ is contained in $\operatorname{Loct}(T)$, and therefore in $\overline{S}$.

Example 3.5.7. Let G a Polish locally compact local group with a left invariant metric, let $\Gamma \subset G$ be a dense local subgroup, and let $\mathcal{H}$ be the minimal pseudogroup generated by the local action of Γ by local left translations on G. The local left and right translations in G by each $g \in G$ will be denoted by L_g and R_g. The restrictions of the local left translations L_γ ($\gamma \in \Gamma$) to open subsets of their domains form a sub-pseudo∗group $S \subset \mathcal{H}$ that generates $\mathcal{H}$. Obviously, $\mathcal{H}$ satisfies with S the condition of being strongly locally free, and therefore strongly quasi-analytic. Moreover $\mathcal{H}$ satisfies with S the condition of being equicontinuous (indeed isometric) by considering any left invariant metric on G. Observe that any local right translation R_g ($g \in G$) generates an equivalence $\mathcal{H} \to \mathcal{H}$.

Now suppose that $\mathcal{H}$ is compactly generated. Then the closure $\overline{\mathcal{H}}$ is generated by the local action of G on itself by local left translations. The sub-pseudo$*$group $\overline{S} \subset \overline{\mathcal{H}}$ consists of the restrictions of the local left translations L_g ($g \in G$) to open subsets of their domains. Observe that $\overline{\mathcal{H}}$ satisfies the condition of being strongly locally free, and therefore strongly quasi-analytic, with $\overline{S}$.

Lemma 3.5.8. *Let G and G' be Polish locally compact local groups with left invariant metrics, let $\Gamma \subset G$ and $\Gamma' \subset G'$ be dense local subgroups, and let $\mathcal{H}$ and $\mathcal{H}'$ be the pseudogroups generated by the local actions of Γ and Γ' by local left translations on G and G'. Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are compactly generated. Then $\mathcal{H}$ and $\mathcal{H}'$ are equivalent if and only if G is locally isomorphic to G'.*

Proof. Consider the notation and observations of Example 3.5.7 for both G and G'; in particular, $S \subset \mathcal{H}$ and $S' \subset \mathcal{H}'$ denote the sub-pseudo$*$groups of restrictions of local translations L_γ and $L_{\gamma'}$ ($\gamma \in \Gamma$ and $\gamma' \in \Gamma'$) to open subsets of their domains. Let e and e' denote the identity elements of G and G'. Let $\Phi : \mathcal{H} \to \mathcal{H}'$ be en equivalence. Since $\mathcal{H}'$ is minimal, after composing Φ with the equivalence generated by some local right translation in G if necessary, we can assume that there is some $\phi \in \Phi$ with $e \in \operatorname{dom}\phi$ and $\phi(e) = e'$.

Let U be a relatively compact open symmetric identity neighborhood in G with $\overline{U} \subset \operatorname{dom}\phi$. Let $\{f_1, \dots, f_n\}$ be a symmetric system of compact generation of $\mathcal{H}$ on U. Thus each f_i has an extension $\tilde{f}_i \in \mathcal{H}$ so that $\overline{\operatorname{dom} f_i} \subset \operatorname{dom}\tilde{f}_i \subset \operatorname{dom}\phi$.

Claim 1. We can assume that $\tilde{f}_i \in S$ and $\phi\tilde{f}_i\phi^{-1} \in S'$ for all i.

Each point in $\operatorname{dom}\tilde{f}_i \cap \operatorname{dom}\phi$ has an open neighborhood O such that $O \subset \operatorname{dom}\tilde{f}_i$, $\tilde{f}_i|_O \in S$ and $\phi\tilde{f}_i\phi^{-1}|_{\phi(O)} \in S'$. Take a finite covering $\{O_{ij}\}$ ($j \in \{1, \dots, k_i\}$) of the compact set $\overline{\operatorname{dom} f_i}$ by sets of this type. Let $\{P_{ij}\}$ be a shrinking of $\{O_{ij}\}$, as cover of $\overline{\operatorname{dom} f_i}$ by open subsets of $\operatorname{dom}\tilde{f}_i$. Then the restrictions $g_{ij} = f_i|_{P_{ij}\cap U}$ ($i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k_i\}$) generate $\mathcal{H}|_U$, each $\tilde{g}_{ij} = \tilde{f}_i|_{O_{ij}}$ is in S and extends g_{ij}, $\overline{\operatorname{dom} g_{ij}} \subset \operatorname{dom}\tilde{g}_{ij}$, and $\phi\tilde{g}_{ij}\phi^{-1} \in S'$, showing Claim 1.

According to Claim 1, the maps $f'_i = \phi f_i \phi^{-1}$ form a symmetric system of compact generation of $\mathcal{H}'$ on $U' = \phi(U)$, which can be checked with the extensions $\tilde{f}'_i = \phi\tilde{f}_i\phi^{-1}$. Let $S_0 \subset S$ and $S'_0 \subset S'$ be the sub-pseudo$*$groups consisting of the restrictions of compositions of maps f_i and f'_i to open subsets of their domains, respectively. They generate $\mathcal{H}$ and $\mathcal{H}'$. It follows from Claim 1 that $\phi f \phi^{-1} \in S'$ for all $f \in S_0$. On the other hand, by Proposition 3.5.3, there is a smaller open identity neighborhood, $V \subset U$, such that, for all $h \in \mathcal{H}$ and $x \in V \cap \operatorname{dom} h$ with $h(x) \in U$, there is some $f \in S_0$ such that $\operatorname{dom} f = V$ and $\boldsymbol{\gamma}(f, x) = \boldsymbol{\gamma}(h, x)$.

Let W be another symmetric open identity neighborhood such that $W^2 \subset V$. Let us show that $\phi : W \to \phi(W)$ is a local isomorphism. Let $\gamma \in W \cap \Gamma$.

The restriction $L_\gamma : W \to \gamma W$ is well defined and belongs to S. Hence there is some $f \in S_0$ so that $\operatorname{dom} f = V$ and $\boldsymbol{\gamma}(f, e) = \boldsymbol{\gamma}(L_\gamma, e)$. Since f is also a restriction of a local left translation in G, it follows that $f = L_\gamma$ on W. So $\phi L_\gamma \phi^{-1}|_{\phi(W)} \in S'$; i.e., there is some $\gamma' \in \Gamma'$ such that $\phi L_\gamma \phi^{-1} = L_{\gamma'}$ on $\phi(W)$. In fact,

$$\phi(\gamma) = \phi L_\gamma(e) = \phi L_\gamma \phi^{-1}(e') = L_{\gamma'}(e') = \gamma' \ .$$

Hence, for all $\gamma, \delta \in \Gamma$,

$$\begin{aligned} \phi(\gamma\delta) &= \phi L_\gamma(\delta) = L_{\phi(\gamma)}\phi(\delta) = \phi(\gamma)\phi(\delta) \ , \\ \phi(\gamma)^{-1} &= L^{-1}_{\phi(\gamma)}(e') = (\phi L_\gamma \phi^{-1})^{-1}(e') \\ &= \phi L_{\gamma^{-1}}\phi^{-1}(e') = L_{\phi(\gamma^{-1})}(e') = \phi(\gamma^{-1}) \ . \end{aligned}$$

Since ϕ, and the product and inversion maps are continuous, it follows that $\phi(gh) = \phi(g)\phi(h)$ and $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g, h \in W$. □

Example 3.5.9. Let G be a locally compact, separable local group with a left-invariant metric, $K \subset G$ a compact subgroup, and $\Gamma \subset G$ a dense sub-local group. The left invariant metric on G can be assumed to be also K-right invariant by the compactness of K. Then the canonical local action of Γ on some neighborhood of the identity class in G/K induces a transitive, equicontinuous and strongly quasi-analytic pseudogroup on a locally compact Polish space; in fact, this is a pseudogroup of local isometries.

Examples 3.5.7 and 3.5.9 are particular cases of pseudogroups induced by local actions (Remark 17) that will play an important role in our theory. For instance, the following result indicates the relevance of Example 3.5.9.

Theorem 3.5.10 (Álvarez-Candel [5, Theorem 5.2])**.** *Let $\mathcal{H}$ be a transitive, compactly generated and equicontinuous pseudogroup on a locally compact Polish space, and suppose that $\overline{\mathcal{H}}$ is strongly quasi-analytic. Then $\mathcal{H}$ is equivalent to a pseudogroup of the type described in Example 3.5.9.*

Remark 24. From the proof of [5, Theorems 3.3 and 5.2], it also follows that, in Theorem 3.5.10, if moreover $\overline{\mathcal{H}}$ is strongly locally free, then $\mathcal{H}$ is equivalent to a pseudogroup of the type described in Example 3.5.7.

Definition 3.5.11. A pseudogroup is called *Riemannian* if it consists of local isometries of a Riemannian manifold.

The condition on pseudogroups to be Riemannian is obviously invariant by pseudogroup equivalences. Clearly, Riemannian pseudogroups are equicontinuous. Assuming transitiveness and compact generation, the following result gives a topological characterization of Riemannian pseudogroups within the class of equicontinuous ones.

Theorem 3.5.12 (Álvarez-Candel [5, Theorem 3.3]). *Let $\mathcal{H}$ be a transitive, compactly generated pseudogroup on a locally compact Polish space T. Then $\mathcal{H}$ is a Riemannian pseudogroup if and only if T is locally connected and finite dimensional, $\mathcal{H}$ is equicontinuous, and $\overline{\mathcal{H}}$ is quasi-analytic.*

3.6 Foliated spaces

Besides pseudogroups generated by local actions, another important example, of a different nature, is the holonomy pseudogroup of a foliated space.

Let X and Z be locally compact Polish spaces. A *foliated chart* in X of *leaf dimension* n, *modeled transversally* on Z, is a pair (U, ϕ), where $U \subseteq X$ is open and $\phi : U \to B \times T$ is a homeomorphism for some open $T \subset Z$ and some open ball B in R^n of finite radius. It is said that U is a *distinguished open set*. The sets $P_y = \phi^{-1}(B \times \{y\})$ $(y \in T)$ are called *plaques* of this foliated chart. For each $x \in B$, the set $S_x = \phi^{-1}(\{x\} \times T)$ is called a *transversal* of the foliated chart. This local product structure defines a local projection $p : U \to T$, called *distinguished submersion*, so that the diagram

$$\begin{tikzcd} U \arrow[rr, "\phi"] \arrow[dr, "p"'] & & B \times T \arrow[dl, "\mathrm{pr}_2"] \\ & T & \end{tikzcd}$$

is commutative.

Let $\mathcal{U} = \{U_i, \phi_i\}$ be a family of foliated charts in X of leaf dimension n modeled transversally on Z and covering X. Assume further that the foliated charts are *coherently foliated* in the sense that, if P and Q are plaques in different charts of $\mathcal{U}$, then $P \cap Q$ is open both in P and Q. Then $\mathcal{U}$ is called a *foliated atlas* on X of *leaf dimension* n and *transversely modeled* on Z. A maximal foliated atlas $\mathcal{F}$ of leaf dimension n and transversely modeled on Z is called a *foliated structure* on X of *leaf dimension* n and *transversely modeled* on Z. Any foliated atlas $\mathcal{U}$ of this type is contained in a unique foliated structure $\mathcal{F}$; then it is said that $\mathcal{U}$ *defines* (or is an atlas of) $\mathcal{F}$. If $Z = \mathbb{R}^m$, then X is a manifold of dimension $n + m$, and $\mathcal{F}$ is traditionally called a *foliation* of *dimension* n and *codimension* m.

For a foliated structure $\mathcal{F}$ on X of dimension n, the plaques form a basis of a topology on X called the *leaf topology*. With the leaf topology, X becomes an n-manifold whose connected components are called *leaves* of $\mathcal{F}$. $\mathcal{F}$ is determined by its leaves.

A foliated atlas $\mathcal{U} = \{U_i, \phi_i\}$ of $\mathcal{F}$ is called *regular* if

(1) each $\overline{U_i}$ is compact subset of a foliated chart (W_i, ψ_i) and $\phi_i = \psi_i|_{U_i}$;

(2) the cover $\{U_i\}$ is *locally finite*; and,

(3) if (U_i, ϕ_i) and (U_j, ϕ_j) are elements of $\mathcal{U}$, then each plaque P of (U_i, ϕ_i) meets at most one plaque of (U_j, ϕ_j).

In this case, for all indices i and j, there is a homeomorphism $h_{ij} : T_{ij} \to T_{ji}$, where $p_i : U_i \to T_i$ is the distinguished submersion defined by (U_i, ϕ_i) and $T_{ij} = p_i(U_i \cap U_j)$, such that the diagram

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ T_{ij} \arrow[rr, "h_{ij}"] & & T_{ji} \end{tikzcd}$$

is commutative. Observe that the cocycle condition $h_{ik} = h_{jk}h_{ij}$ is satisfied on $T_{ijk} = p_i(U_i \cap U_j \cap U_k)$. For this reason, $\{U_i, p_i, h_{ij}\}$ is called a *defining cocycle* of $\mathcal{F}$ with values in Z. The equivalence class of the pseudogroup $\mathcal{H}$ generated by the maps h_{ij} on $T = \bigsqcup_{i\in I} T_i$ is called the *holonomy pseudogroup* of the foliated space $(X, \mathcal{F})$; $\mathcal{H}$ is the representative of the holonomy pseudogroup of $(X, \mathcal{F})$ induced by the defining cocycle $\{U_i, p_i, h_{ij}\}$. This T can be identified with a *total* (or *complete*) *transversal* to the leaves in the sense that it meets all leaves and is locally given by the transversals defined by foliated charts. All compositions of a finite number of maps h_{ij} form a pseudo$*$group S that generates $\mathcal{H}$, called the *holonomy pseudo$*$group* of $\mathcal{F}$ induced by $\{U_i, p_i, h_{ij}\}$. There is a canonical identity between the space of leaves and the space of $\mathcal{H}$-orbits, $X/\mathcal{F} \equiv T/\mathcal{H}$.

A foliated atlas (respectively, defining cocycle) contained in another one is called *sub-foliated atlas* (respectively, *sub-foliated cocycle*).

The *holonomy group* of each leaf L is defined as the germ group of the corresponding orbit. It can be considered as a quotient of $\pi_1(L)$ by taking "chains" of sets U_i along loops in L; this representation of $\pi_1(L)$ is called the *holonomy representation.* The kernel of the holonomy representation is equal to $q_*\pi_1(\widetilde{L})$ for a regular covering space $q : \widetilde{L} \to L$, which is called the *holonomy cover* of L. If $\mathcal{F}$ admits a countable defining cocycle, then the leaves in some dense G_δ subset of M have trivial holonomy groups [26, 12], and therefore they can be identified with their holonomy covers.

It is said that a foliated space is (*topologically*) *transitive* or *minimal* if any representative of its holonomy pseudogroup is such. Transitiviness (respectively, minimality) of a foliated space means that some leaf is dense (respectively, all leaves are dense).

Haefliger [25] has observed that, if X is compact, then $\mathcal{H}$ is compactly generated, which can be seen as follows. There is some defining cocycle $\{U'_i, p'_i, h'_{ij}\}$, with $p'_i : U'_i \to T'_i$, such that $\overline{U_i} \subset U'_i$, $T_i \subset T'_i$ and p'_i extends p_i.

Therefore each h'_{ij} is an extension of h_{ij} so that $\overline{\operatorname{dom} h_{ij}} \subset \operatorname{dom} h'_{ij}$. Moreover $\mathcal{H}$ is the restriction to T of the pseudogroup $\mathcal{H}'$ on $T' = \bigsqcup_i T'_i$ generated by the maps h'_{ij}, and T is a relatively compact open subset of T' that meets all $\mathcal{H}'$-orbits.

3.7 Equicontinuous foliated spaces

Definition 3.7.1. It is said that a foliated space is *equicontinuous* if any representative of its holonomy pseudogroup is such.

Remark 25. The definition of equicontinuity for a foliated space makes sense by Lemma 3.5.2.

Examples of equicontinuous foliated spaces will be given in Chapter 7.

Definition 3.7.2. A foliation is called *Riemannian* if any representative of its holonomy pseudogroup is Riemannian.

Remark 26. A Riemannian foliation is "transversely smooth" by definition, but "tangential smoothness" is not required; thus the ambient manifold may not be smooth.

The following theorem is the main theorem of Álvarez and Candel in [5]. It gives a topological characterization of transitive Riemannian foliations within the wider class of equicontinuous foliated spaces. It is a direct consequence of Theorem 3.5.12.

Theorem 3.7.3. *Let $(X, \mathcal{F})$ be a transitive compact foliated space. Then $\mathcal{F}$ is a Riemannian foliation if and only if X is locally connected and finite dimensional, $\mathcal{F}$ is equicontinuous, and the closure of its holonomy pseudogroup is quasi-analytic.*

3.8 Coarse quasi-isometries and growth of metric spaces

A *net* in a metric space M, with metric d, is a subset $A \subset M$ that satisfies $d(x, A) \leq C$ for some $C > 0$ and all $x \in M$; the term C-*net* is also used. A *coarse quasi-isometry* between M and another metric space M' is a bi-Lipschitz bijection between nets of M and M'; in this case, M and M' are said to be *coarsely quasi-isometric* (in the sense of Gromov) [21]. If such a bi-Lipschitz bijection, as well as its inverse, has dilation $\leq \lambda$, and it is defined between C-nets, then it will be said that the coarse quasi-isometry has *distortion* (C, λ). A family of coarse quasi-isometries with a common distortion will be called *uniform*, and the corresponding metric spaces are called *uniformly* coarsely quasi-isometric.

The version of growth for metric spaces given here is taken from [6]. Since [6] is not finished, some short proofs are included.

Recall that, given non-decreasing functions[5] $u, v : [0,\infty) \to [0,\infty)$, it is said that u is *dominated* by v, written $u \preccurlyeq v$, when there are $a, b \geq 1$ and $c \geq 0$ such that $u(r) \leq a\, v(br)$ for all $r \geq c$. If $u \preccurlyeq v \preccurlyeq u$, then it is said that u and v represent the same *growth type*; this is an equivalence relation and "$\preccurlyeq$" defines a partial order relation between growth types called *domination*. For a family of pairs of non-decreasing functions $[0,\infty) \to [0,\infty)$, *uniform domination* means that those pairs satisfy the above condition of domination with the same constants a, b, c. A family of functions $[0,\infty) \to [0,\infty)$ will be said to have *uniformly* the same growth type if they uniformly dominate one another.

For a complete connected Riemannian manifold L, the growth type of each mapping $r \mapsto \operatorname{vol} B(x,r)$ is independent of x and is called the *growth type* of L. Another definition of *growth type* can be similarly given for metric spaces whose bounded sets are finite, where the number of points is used instead of the volume.

Let M be a metric space with metric d. A *quasi-lattice* Γ of M is a C-net of M for some $C \geq 0$ such that, for every $r \geq 0$, there is some $K_r \geq 0$ such that $\operatorname{card}(\Gamma \cap B(x,r)) \leq K_r$ for every $x \in M$. It is said that M is of *coarse bounded geometry* if it has a quasi-lattice. In this case, the *growth type* of M can be defined as the growth type of any quasi-lattice Γ of M; i.e., it is the growth type of the *growth function* $r \mapsto v_\Gamma(x,r) = \operatorname{card}(B(x,r) \cap \Gamma)$ for any $x \in \Gamma$. This definition can be proved to be independent of Γ in the following way. Let Γ' be another quasi-lattice in M. So Γ and Γ' are C-nets in M for some $C \geq 0$, and there is some $K_r \geq 0$ for each $r \geq 0$ such that $\operatorname{card}(B(x,r) \cap \Gamma) \leq K_r$ and $\operatorname{card}(B(x,r) \cap \Gamma') \leq K_r$ for all $x \in M$. Fix points $x \in \Gamma$ and $x' \in \Gamma'$, and let $\delta = d(x,x')$. Because $B(x,r) \subset B(x', r+\delta)$ and Γ' is a C-net, it follows that

$$B(x,r) \cap \Gamma \subset \bigcup_{y' \in B(x', r+\delta+C) \cap \Gamma'} B(y', C) \cap \Gamma' ,$$

yielding

$$v_\Gamma(x,r) \leq K_C\, v_{\Gamma'}(x', r+\delta+C) \leq K_C\, v_{\Gamma'}(x', (1+\delta+C)r)$$

for all $r \geq 1$. Hence the growth type of $r \mapsto v_\Gamma(x,r)$ is dominated by the growth type of $r \mapsto v_{\Gamma'}(x,r)$.

[5]Usually, growth types are defined by using non-decreasing functions $\mathbb{Z}^+ \to [0,\infty)$, but this gives rise to an equivalent concept.

For a family of metric spaces, if they satisfy the above condition of coarse bounded geometry with the same constants C and K_r, then they are said to have *uniformly* coarse bounded geometry. If moreover the lattices involved in this condition have growth functions defining uniformly the same growth type, then these metric spaces are said to have *uniformly* the same growth type.

The condition of coarse bounded geometry is satisfied by complete connected Riemannian manifolds of bounded geometry, and by discrete metric spaces with a uniform upper bound on the number of points in all balls of each given radius [10]. In those cases, the two given definitions of growth type are equal.

Lemma 3.8.1 (Álvarez-Candel [4])**.** *Two coarsely quasi-isometric metric spaces of coarse bounded geometry have the same growth type. Moreover, if a family of metric spaces are uniformly coarsely quasi-isometric to each other, then they have uniformly the same growth type.*

Proof. Let $\phi : A \to A'$ be a coarse quasi-isometry between metric spaces M and M' of coarse bounded geometry. Then A is of coarse bounded geometry too, and thus it has some lattice Γ, which is also a lattice in M because A is a net. Since ϕ is a bi-Lipschitz bijection, it easily follows that Γ and $\phi(\Gamma)$ have the same growth type, and that $\phi(\Gamma)$ is a lattice in A', and thus in M' too because A' is a net. This argument has an obvious uniform version for a family of metric spaces. □

3.9 Quasi-isometry type of orbits

Let $\mathcal{H}$ be a pseudogroup on a space T, and E a symmetric set of generators of $\mathcal{H}$. Let $\mathfrak{G}$ be the groupoid of germs of maps in $\mathcal{H}$.

For each $h \in \mathcal{H}$ and $x \in \operatorname{dom} h$, let $|h|_{E,x}$ be the length of the shortest expression of $\boldsymbol{\gamma}(h, x)$ as product of germs of maps in E (being 0 if $\boldsymbol{\gamma}(h, x) = \boldsymbol{\gamma}(\mathrm{id}_T, x)$). For each $x \in T$, define metrics d_E on $\mathcal{H}(x)$ and $\mathfrak{G}_x$ by

$$d_E(y, z) = \min\{\, |h|_{E,x} \mid h \in \mathcal{H},\ y \in \operatorname{dom} h,\ h(y) = z \,\} ,$$
$$d_E(\boldsymbol{\gamma}(f, x), \boldsymbol{\gamma}(g, x)) = |fg^{-1}|_{E,g(x)} .$$

Notice that

$$d_E(f(x), g(x)) \le d_E(\boldsymbol{\gamma}(f, x), \boldsymbol{\gamma}(g, x)) .$$

Moreover, on the germ covers, d_E is right invariant in the sense that, if $y \in \mathcal{H}(x)$, the bijection $\mathfrak{G}_y \to \mathfrak{G}_x$, given by right multiplication with any element in $\mathfrak{G}_x^y$, is isometric; so the isometry types of the germ covers of the orbits make sense without any reference to base points. In fact, the definition of d_E on

$\mathfrak{G}_x$ is analogous to the right invariant metric d_S on a group Γ induced by a symmetric set of generators S: $d_S(\gamma,\delta)=|\gamma\delta^{-1}|$ for $\gamma,\delta\in\Gamma$, where $|\gamma|$ is the length of the shortest expression of γ as product of elements of S (being 0 if $\gamma=e$).

Assume that $\mathcal{H}$ is compactly generated and T locally compact. Let $U\subset T$ be a relatively compact open subset that meets all $\mathcal{H}$-orbits, let $\mathcal{G}=\mathcal{H}|_U$, and let E be a symmetric system of compact generation of $\mathcal{H}$ on U. With this conditions, the quasi-isometry type of the $\mathcal{G}$-orbits with d_E may depend on E [4, Section 6]. So the following additional condition on E is considered.

Definition 3.9.1 (Álvarez-Candel [4, Definition 4.2]). With the above notation, it is said that E is *recurrent* if, for any relatively compact open subset $V\subset U$ that meets all $\mathcal{G}$-orbits, there exists some $R>0$ such that $\mathcal{G}(x)\cap V$ is an R-net in $\mathcal{G}(x)$ with d_E for all $x\in U$.

The role played by V in Definition 3.9.1 can actually be played by any relatively compact open subset that meets all orbits [4, Lemma 4.3]. Furthermore there always exists a recurrent system of compact generation on U [4, Corollary 4.5].

Theorem 3.9.2 (Álvarez-Candel [4, Theorem 4.6]). *Let $\mathcal{H}$ and $\mathcal{H}'$ be compactly generated pseudogroups on locally compact spaces T and T', let U and U' be relatively compact open subsets of T and T' that meet all orbits of $\mathcal{H}$ and $\mathcal{H}'$, let $\mathcal{G}$ and $\mathcal{G}'$ denote the restrictions of $\mathcal{H}$ and $\mathcal{H}'$ to U and U', and let E and E' be recurrent symmetric systems of compact generation of $\mathcal{H}$ and $\mathcal{H}'$ on U and U', respectively. Suppose that there exists an equivalence $\mathcal{H}\to\mathcal{H}'$, and consider the induced equivalence $\mathcal{G}\to\mathcal{G}'$ and homeomorphism $U/\mathcal{G}\to U'/\mathcal{G}'$. Then the $\mathcal{G}$-orbits with d_E are uniformly quasi-isometric to the corresponding $\mathcal{G}'$-orbits with $d_{E'}$.*

An obvious modification of the arguments of the proof of [4, Theorem 4.6] gives the following.

Theorem 3.9.3. *With the notation and conditions of Theorem 3.9.2, the germ covers of the $\mathcal{G}$-orbits with d_E are uniformly quasi-isometric to the germ covers of the corresponding $\mathcal{G}'$-orbits with $d_{E'}$.*

Corollary 3.9.4. *With the notation and conditions of Theorems 3.9.2 and 3.9.3, the corresponding orbits of $\mathcal{G}$ and $\mathcal{G}'$, as well as their germ covers, have the same growth type, uniformly.*

Proof. This follows from Lemma 3.8.1 and Theorems 3.9.2 and 3.9.3. □

Example 3.9.5. Let G be a locally compact Polish local group with a left-invariant metric, let $\Gamma\subset G$ be a dense finitely generated sub-local group, and let $\mathcal{H}$ denote the pseudogroup generated by the local action of Γ on G by local left translations. Suppose that $\mathcal{H}$ is compactly generated, and let $\mathcal{G} = \mathcal{H}|_U$ for some relatively compact open identity neighborhood U in G, which meets all $\mathcal{H}$-orbits because Γ is dense. For every $\gamma \in \Gamma$ with $\gamma U \cap U \neq \emptyset$, let h_γ denote the restriction $U \cap \gamma^{-1}U \to \gamma U \cap U$ of the local left translation by γ. There is a finite symmetric set $S = \{s_1, \dots, s_k\} \subset \Gamma$ such that $E = \{h_{s_1}, \dots, h_{s_k}\}$ is a recurrent system of compact generation of $\mathcal{H}$ on U; in fact, by reducing Γ if necessary, we can assume that S generates Γ. The recurrence of E means that there is some $N \in \mathbb{N}$ such that

$$U = \bigcup_{h \in E^N} h^{-1}(V \cap \operatorname{im} h)\ , \tag{3.1}$$

where E^N is the family of compositions of at most N elements of E.

For each $x \in U$, let

$$\Gamma_{U,x} = \{\, \gamma \in \Gamma \mid \gamma x \in U \,\}\ .$$

Let $\mathfrak{G}$ denote the topological groupoid of germs of $\mathcal{G}$. The map $\Gamma_{U,x} \to \mathfrak{G}_x$, $\gamma \mapsto \boldsymbol{\gamma}(h_\gamma, x)$, is bijective. For $\gamma \in \Gamma_{U,x}$, let $|\gamma|_{S,U,x} := |h_\gamma|_{E,x}$. Thus $|e|_{S,U,x} = 0$, and, if $\gamma \neq e$, then $|\gamma|_{S,U,x}$ equals the minimum $n \in \mathbb{N}$ such that there are $i_1, \dots, i_n \in \{1, \dots, k\}$ with $\gamma = s_{i_n} \cdots s_{i_1}$ and $s_{i_m} \cdots s_{i_1} \cdot x \in U$ for all $m \in \{1, \dots, n\}$. Moreover d_E on $\mathfrak{G}_x$ corresponds to the metric $d_{S,U,x}$ on $\Gamma_{U,x}$ given by

$$d_{S,U,x}(\gamma, \delta) = |\delta\gamma^{-1}|_{S,U,\gamma(x)}\ .$$

Observe that, for all $\gamma \in \Gamma_{U,x}$ and $\delta \in \Gamma_{U,\gamma\cdot x}$,

$$\delta\gamma \in \Gamma_{U,x}\ , \qquad |\delta\gamma|_{S,U,x} \leq |\gamma|_{S,U,x} + |\delta|_{S,U,\gamma\cdot x}\ , \tag{3.2}$$

$$\gamma^{-1} \in \Gamma_{U,\gamma\cdot x}\ , \qquad |\gamma|_{S,U,x} = |\gamma^{-1}|_{S,U,\gamma\cdot x}\ . \tag{3.3}$$

3.10 Growth of leaves

Let X be a compact Polish foliated space whose foliated structure $\mathcal{F}$ is given by a defining cocycle $\{U_i, p_i, h_{ij}\}$ induced by a regular foliation atlas, where $p_i : U_i \to T_i$ and $h_{ij} : T_{ij} \to T_{ji}$. As we saw in Section 3.6, $\mathcal{H}$ can be considered as the restriction of some compactly generated pseudogroup $\mathcal{H}'$ to some relatively compact open subset, and $E = \{h_{ij}\}$ is a system of compact generation on T. Moreover Álvarez and Candel [4] observed that E is recurrent. According to Theorems 3.9.2 and 3.9.3, it follows that the quasi-isometry type of the $\mathcal{H}$-orbits and their germ covers with d_E are independent of the choice of $\{U_i, p_i, h_{ij}\}$ under the above conditions; thus they can be considered as quasi-isometry types of the corresponding leaves and they holonomy covers.

This has the following interpretation when X is a smooth manifold. In this case, given any Riemannian metric g on X, for each leaf L, the differentiable (and coarse) quasi-isometry type of $g|_L$ and its lift to $\widetilde{L}$ are independent of the choice of g; they depend only on $\mathcal{F}$ and L; in fact, they are coarsely quasi-isometric to the corresponding leaves, and therefore they have the same growth type [13] (this is an easy consequence of the existence of a uniform bound of the diameter of the plaques). Similarly, the germ covers of the $\mathcal{H}$-orbits are also quasi-isometric to the holonomy covers of the corresponding leaves.

3.11 Topological Tits alternative

The following result follows from the arguments used to prove one of the cases of the topological Tits alternative of Breuillard-Gelander. It was the key tool used by these authors, together with Molino's structure theorem, to study the growth of Riemannian foliations.

Proposition 3.11.1 (Breuillard-Gelander [11, Proposition 10.5]). *Let G be a non-nilpotent connected real Lie group and Γ a finitely generated dense subgroup. For any finite set $S = \{s_1, \dots, s_k\}$ of generators of Γ, and any neighborhood B of e in G, there are elements $t_i \in \Gamma \cap s_i B$ ($i \in \{1, \dots, k\}$) which freely generate a free semi-group. If G is not solvable, then we can choose the elements t_i so that they freely generate a free group.*

Chapter 4

Molino's theory for equicontinuous pseudogroups

Contents

4.1 Conditions on $\mathcal{H}$

Let T be a locally compact Polish space. Let $\mathcal{H}$ be a pseudogroup of local transformations of T. Suppose that $\mathcal{H}$ is compactly generated, complete and equicontinuous, and that $\overline{\mathcal{H}}$ is also strongly-quasi-analytic.

Let U be a relatively compact open set in T that meets all orbits of $\mathcal{H}$. The condition of compact generation is satisfied with U. Consider a representative $\{T_i, d_i\}$ of a quasi-local metric on T satisfying the condition of equicontinuity of $\mathcal{H}$ with some sub-pseudo$*$group $S \subset \mathcal{H}$ that generates $\mathcal{H}$. We can also suppose that S satisfies the condition of strong quasi-analyticity of $\mathcal{H}$.

Remark 27. Consider the sub-pseudo$*$group $\overline{S} \subset \overline{\mathcal{H}}$ defined in Remark 22 (page 30). According to Theorem 3.5.6 and Remark 22, there is a mapping $\varepsilon \mapsto \delta(\varepsilon) > 0$, for $\varepsilon > 0$, such that

$$d_i(x, y) < \delta(\varepsilon) \Longrightarrow d_j(h(x), h(y)) < \varepsilon$$

for all indices i and j, every $h \in \overline{S}$, and $x, y \in T_i \cap h^{-1}(T_j \cap \operatorname{im} h)$.

Remark 28. By Remark 23 and refining $\{T_i\}$ if necessary, we can assume that $\overline{U}$ is covered by a finite collection of the sets T_i, $\{T_{i_1}, \dots, T_{i_r}\}$, such that the closure of

$$\{\, h \in C(T_{i_k}, T) \cap S \mid h(T_{i_k}) \cap \overline{U} \neq \emptyset \,\}$$

in $C_{\text{c-o}}(T_{i_k}, T)$ is contained in $\overline{S}$ for all $k \in \{1, \dots, r\}$.

Remark 29. By Proposition 3.5.3 and Remark 28, and refining $\{T_i\}$ if necessary, we can assume that, for all $h \in \overline{\mathcal{H}}$ and $x \in T_{i_k} \cap U \cap \operatorname{dom} h$ with $h(x) \in U$, there is some $\tilde{h} \in \overline{S}$ with $\operatorname{dom} \tilde{h} = T_{i_k}$ and $\boldsymbol{\gamma}(h, x) = \boldsymbol{\gamma}(\tilde{h}, x)$.

Remark 30. By Remarks 6, 11 and 20, and refining $\{T_i\}$ if necessary, we can assume that the strong quasi-analyticity of $\overline{\mathcal{H}}$ is satisfied with $\overline{S}$.

4.2 Coincidence of topologies

The proof of the following result is inspired by [3].

Proposition 4.2.1. $\overline{S}_{\text{b-c-o}} = \overline{S}_{\text{c-o}}$.

Proof. For each $g \in \mathcal{H}$, take any index i and open sets $V, W \subset T$ such that $\overline{V} \subset W$ and $\overline{W} \subset \operatorname{im} g$. By Proposition 3.5.4, there is some $\varepsilon(i, V, W) > 0$ such that, for all $x, y \in T_i$, if $d_i(x, y) < \varepsilon(i, V, W)$, then

$$f(x) \in \overline{V} \Longrightarrow f(y) \in W$$

for all $f \in \overline{S}$ with $x, y \in \operatorname{dom} f$. Let $\mathcal{K}(g, i, V, W)$ be the family of compact subsets $K \subset T_i \cap \operatorname{dom} g$ such that

$$\mathring{K} \neq \emptyset\,, \quad \operatorname{diam}_{d_i}(K) < \varepsilon(i, V, W)\,, \quad g(K) \subset V\,,$$

where $\mathring{K}$ and $\operatorname{diam}_{d_i}(K)$ denotes the interior and d_i-diameter of K. Moreover let $\mathcal{K}(g)$ denote the union of the families $\mathcal{K}(g, i, V, W)$ as above. Then a subbasis $\mathcal{N}(g)$ of open neighborhoods of each g in $\overline{\mathcal{H}}_{\text{c-o}}$ is given by the sets $\mathcal{N}(K, O) \cap \overline{S}$, where $K \in \mathcal{K}(g)$ and O is an open subset of T such that $g(K) \subset O$.

We have to prove the continuity of the inversion map $\overline{S}_{\text{c-o}} \to \overline{S}_{\text{c-o}}$, $h \mapsto h^{-1}$. Let $h \in \overline{S}$ and let $\mathcal{N}(K, O) \in \mathcal{N}(h^{-1})$ with $K \in \mathcal{K}(h^{-1}, i, V, W)$, and fix any point $x \in \mathring{K}$. Then

$$\mathcal{V} = \mathcal{N}(\{h^{-1}(x)\}, \mathring{K}) \cap \mathcal{N}(\overline{W} \setminus O, T \setminus K)$$

is an open neighborhood of h in $\mathcal{H}_{\text{c-o}}$. For any $f \in \mathcal{V} \cap \overline{S}$ and $y \in K$, we have $d_i(fh^{-1}(x), y) < \varepsilon(i, V, W)$ because $fh^{-1}(x) \in \mathring{K}$ and $\operatorname{diam}_{d_i}(K) < \varepsilon(i, V, W)$. So $f^{-1}(y) \in W$ by the definition of $\varepsilon(i, V, W)$ since $f^{-1} \in \overline{S}$ and $h^{-1}(x) \in h^{-1}(K) \subset V$. Therefore, if $f^{-1}(y) \notin O$, we get $f^{-1}(y) \in \overline{W} \setminus O$, obtaining $y \in T \setminus K$, which is a contradiction. Hence $f^{-1} \in \mathcal{N}(K, O)$ for all $f \in \mathcal{V} \cap \overline{S}$. □

Let $\overline{\mathfrak{G}}$ denote the groupoid of germs of $\overline{\mathcal{H}}$. The following direct consequence of Proposition 4.2.1 gives a partial answer to Question 3.3.7.

Corollary 4.2.2. $\overline{\mathfrak{G}}_{\overline{S},\text{b-c-o}} = \overline{\mathfrak{G}}_{\overline{S},\text{c-o}}$.

Thus $\overline{\mathfrak{G}}_{\overline{S},\text{c-o}}$ is a topological groupoid by Corollary 4.2.2.

4.3 The space $\widehat{T}$

Recall that $s, t : \overline{\mathfrak{G}}_{\overline{S},\text{c-o}} \to T$ denote the source and target projections. Let $\widehat{T} = \mathfrak{G}_{\overline{S},\text{c-o}}$, where the following subsets are open:

$$\begin{aligned}\widehat{T}_U &= s^{-1}(U) \cap t^{-1}(U) \ ,\\ \widehat{T}_{k,l} &= s^{-1}(T_{i_k,i_l}) \cap t^{-1}(T_{i_k,i_l}) \ ,\\ \widehat{T}_{U,k,l} &= \widehat{T}_U \cap \widehat{T}_{k,l} \ .\end{aligned}$$

Observe that $\widehat{T}_U$ is an open subspace of $\widehat{T}$, and the family of sets $\widehat{T}_{U,k,l}$ form an open covering of $\widehat{T}_U$.

Let $\boldsymbol{\gamma}(h, x) \in \widehat{T}_{U,k,l}$. We can assume that $h \in \overline{S}$ and $\operatorname{dom} h = T_{i_k}$ according to Remark 29. Since $x \in T_{i_k} \cap U$ and $h(x) \in T_{i_l} \cap U$, there are relatively compact open neighborhoods, V of x and W of $h(x)$, such that $\overline{V} \subset T_{i_k} \cap U$, $\overline{W} \subset T_{i_l} \cap U$ and $h(\overline{V}) \subset W$.

By Remark 29, for each $f \in \overline{S}$ with $x \in \operatorname{dom} f$, there is some $\tilde{f} \in \overline{S}$ with $\operatorname{dom} \tilde{f} = T_{i_k}$ and $\boldsymbol{\gamma}(\tilde{f}, x) = \boldsymbol{\gamma}(f, x)$.

Lemma 4.3.1. *$f = \tilde{f}$ on V.*

Proof. The composition $f|_V \, \tilde{f}^{-1}$ is defined on $\tilde{f}(V)$, belongs to $\overline{S}$, and is the identity on some neigborhood of $\tilde{f}(x) = f(x)$. So $f|_V \, \tilde{f}^{-1}$ is the identity on $\tilde{f}(V)$ because $\overline{\mathcal{H}}$ satisfies the strong quasi-analyticity condition with $\overline{S}$. Hence $f = \tilde{f}$ on V. □

Let

$$\overline{S}_0 = \{\, f \in \overline{S} \mid \overline{V} \subset \operatorname{dom} f,\ f(\overline{V}) \subset W \,\} \ , \tag{4.1}$$

$$\overline{S}_1 = \{\, f \in \overline{S} \mid \overline{V} \subset \operatorname{dom} f,\ f(\overline{V}) \subset \overline{W} \,\} \ , \tag{4.2}$$

equipped with the restriction of the compact-open topology. Notice that $\overline{S}_0$ is an open neighborhood of h in $\overline{S}_{\text{c-o}}$. Consider the compact-open topology on $C(\overline{V}, \overline{W})$.

Lemma 4.3.2. *The restriction map $\mathcal{R} : \overline{S}_1 \to C(\overline{V}, \overline{W})$, $\mathcal{R}(f) = f|_{\overline{V}}$, defines an identification $\mathcal{R} : \overline{S}_1 \to \mathcal{R}(\overline{S}_1)$.*

Proof. The continuity of $\mathcal{R}$ is elementary.

Let $G \subset \mathcal{R}(\overline{S}_1)$ such that $\mathcal{R}^{-1}(G)$ is open in $\overline{S}_1$. For each $g_0 \in G$, there is some $g_0' \in \mathcal{R}^{-1}(G)$ such that $\mathcal{R}(g_0') = g_0$. Since $\mathcal{R}^{-1}(G)$ is open in $\overline{S}_1$, there are finite collections, $\{K_1, \dots, K_p\}$ of compact subsets and $\{O_1, \dots, O_p\}$ of open subsets, such that

$$\begin{aligned}g_0' \in \{\, f \in \overline{S}_1 \mid K_1 \cup \dots \cup K_p \subset \operatorname{dom} f,&\\ f(K_1) \subset O_1, \dots, f(K_p) \subset O_p \,\} \subset \mathcal{R}^{-1}(G) \ .&\end{aligned}$$

Then

$$g_0 \in \{\, g \in \overline{S}_1 \mid (K_1 \cup \cdots \cup K_p) \cap \overline{V} \subset \operatorname{dom} g,\\ g(K_1 \cap \overline{V}) \subset O_1 \cap \overline{W}, \ldots, g(K_p \cap \overline{V}) \subset O_p \cap \overline{W} \,\} \subset G\,.$$

Since $K_1 \cap \overline{V}, \ldots, K_p \cap \overline{V}$ are compact in $\overline{V}$, and $O_1 \cap \overline{W}, \ldots, O_p \cap \overline{W}$ are open in $\overline{W}$, it follows that g_0 is in the interior of G in $\mathcal{R}(\overline{S}_1)$. Hence G is open in $\mathcal{R}(\overline{S}_1)$. □

Lemma 4.3.3. *$\mathcal{R}(\overline{S}_1)$ is closed in $C(\overline{V}, \overline{W})$.*

Proof. Observe that $C(\overline{V}, \overline{W})$ is second countable because T is Polish. Take a sequence g_n in $\mathcal{R}(\overline{S}_1)$ converging to g in $C(\overline{V}, \overline{W})$. Then it easily follows that $g_n|_V$ converges to $g|_V$ in $C(V, T)$ with the compact-open topology. Thus $g|_V \in \overline{S}$ according to Remark 28, and let $f = \widetilde{g|_V}$. By Lemma 4.3.1, we have $g = f|_{\overline{V}}$. Therefore $f \in \overline{S}_1$ and $g = \mathcal{R}(f)$. □

Corollary 4.3.4. *$\mathcal{R}(\overline{S}_1)$ is compact in $C(\overline{V}, \overline{W})$.*

Proof. This follows by Arzela-Ascoli Theorem and Lemma 4.3.3 because $\overline{V}$ and $\overline{W}$ are compact, and $\mathcal{R}(\overline{S}_1)$ is equicontinuous since $\overline{\mathcal{H}}$ satisfies the equicontinuity condition with $\overline{S}$ and $\{T_i, d_i\}$. □

Let V_0 be an open subset of T such that $x \in V_0$ and $\overline{V_0} \subset V$. Since $\overline{V_0} \subset \operatorname{dom} f$ for all $f \in \overline{S}_1$, we can consider the restriction $\overline{S}_1 \times \overline{V_0} \to \widehat{T}$ of the germ map.

Lemma 4.3.5. *$\boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0})$ is compact in $\widehat{T}$.*

Proof. For each $g \in C(\overline{V}, \overline{W})$ and $y \in \overline{V}$, let $\overline{\boldsymbol{\gamma}}(g, y)$ denote the germ of g at y, defining a germ map

$$\overline{\boldsymbol{\gamma}} : C(\overline{V}, \overline{W}) \times \overline{V} \to \overline{\boldsymbol{\gamma}}(C(\overline{V}, \overline{W}) \times \overline{V})\,.$$

Since $\overline{V_0} \subset V$, we get that $\boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0}) = \overline{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0})$ and the diagram

$$\begin{array}{ccc} \overline{S}_1 \times \overline{V_0} & \xrightarrow{\mathcal{R}\times\mathrm{id}} & \mathcal{R}(\overline{S}_1) \times \overline{V_0} \\ {\scriptstyle\boldsymbol{\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\boldsymbol{\gamma}}} \\ \boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0}) & = & \overline{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0}) \end{array} \tag{4.3}$$

is commutative. Then

$$\overline{\boldsymbol{\gamma}} : \mathcal{R}(\overline{S}_1) \times \overline{V_0} \to \overline{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0})$$

is continuous because

$$\mathcal{R} \times \mathrm{id} : \overline{S}_1 \times \overline{V_0} \to \mathcal{R}(\overline{S}_1) \times \overline{V_0}$$

is an identification by Lemma 4.3.2 and

$$\boldsymbol{\gamma} : \overline{S}_1 \times \overline{V_0} \to \boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0})$$

is continuous. Hence $\boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0})$ is compact by Corollary 4.3.4. □

Lemma 4.3.6. $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ *is open in* $\widehat{T}$.

Proof. This follows because $\overline{S}_0 \times V_0$ is open in $\overline{S}_{\text{c-o}} * T$ and saturated by the fibers of $\boldsymbol{\gamma} : \overline{S}_{\text{c-o}} * T \to \widehat{T}$. □

Remark 31. Observe that the proof of Lemma 4.3.6 does not require $\overline{V_0} \subset V$; it holds for any open $V_0 \subset V$.

Corollary 4.3.7. $\widehat{T}_U$ *is locally compact.*

Proof. We have that $\boldsymbol{\gamma}(\overline{S}_0 \times \overline{V_0})$ is compact by Lemma 4.3.5 and contains $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$, which is an open neighborhood of $\boldsymbol{\gamma}(h, x)$ by Lemma 4.3.6. Then the result follows because $\boldsymbol{\gamma}(h, x) \in \widehat{T}_U$ is arbitrary. □

Lemma 4.3.8. $\overline{\boldsymbol{\gamma}} : \mathcal{R}(\overline{S}_1) \times \overline{V_0} \to \widehat{T}$ *is injective.*

Proof. Let

$$(\mathcal{R}(f_1), y_1), (\mathcal{R}(f_2), y_2) \in \mathcal{R}(\overline{S}_1) \times \overline{V_0}$$

for $f_1, f_2 \in \overline{S}_1$ with $\overline{\boldsymbol{\gamma}}(\mathcal{R}(f_1), y_1) = \overline{\boldsymbol{\gamma}}(\mathcal{R}(f_2), y_2)$. Thus $y_1 = y_2 =: y$ and $\boldsymbol{\gamma}(f_1, y_1) = \boldsymbol{\gamma}(f_2, y_2)$; i.e., $f_1 = f_2$ on some neighborhood O of y in $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Then $f_1(O) \subset \operatorname{dom}(f_1 f_2^{-1})$, and $f_2 f_1^{-1} = \mathrm{id}_T$ on $f_1(O)$. Since $f_2 f_1^{-1} \in \overline{S}$, we get $f_2 f_1^{-1} = \mathrm{id}_T$ on $\operatorname{dom}(f_2 f_1^{-1}) = f_1(\operatorname{dom} f_1 \cap \operatorname{dom} f_2)$ by the strong quasi-analyticity of $\overline{S}$. Since $\overline{V} \subset \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, it follows that $f_2 f_1^{-1} = \mathrm{id}_T$ on $f_1(\overline{V})$, and therefore $f_1 = f_2$ on $\overline{V}$; i.e., $\mathcal{R}(f_1) = \mathcal{R}(f_2)$. □

Let $\hat{\pi} := (s, t) : \widehat{T} \to T \times T$, which is continuous.

Corollary 4.3.9. *The restriction* $\hat{\pi} : \widehat{T}_U \to U \times U$ *is proper.*

Proof. Since $U \times U$ can be covered by sets of the form $V_0 \times W$, for V_0 and W as above, it is enough to prove that $\hat{\pi}^{-1}(K_1 \times K_2)$ is compact for all compact set $K_1 \subset V_0$ and $K_2 \subset W$. But then, with the above notation, we obtain

$$\hat{\pi}^{-1}(K_1 \times K_2) \subset \boldsymbol{\gamma}(\overline{S}_1 \times K_1) \subset \boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0}) \ ,$$

and the result follows from Lemma 4.3.5. □

Corollary 4.3.10. *The closure of* $\widehat{T}_U$ *in* $\widehat{T}$ *is compact.*

Proof. Take a relatively compact open subset $U' \subset T$ containing $\overline{U}$. By applying Corollary 4.3.9 to U', it follows that $\hat{\pi} : \widehat{T}_{U'} \to U' \times U'$ is proper. Therefore $\hat{\pi}^{-1}(\overline{U} \times \overline{U})$ is compact and contains the closure of $\widehat{T}_U$ in $\widehat{T}$. □

Lemma 4.3.11. *$\widehat{T}_U$ is Hausdorff.*

Proof. Let $\boldsymbol{\gamma}(h_1, x_1) \neq \boldsymbol{\gamma}(h_2, x_2)$ in $\widehat{T}_U$.

Suppose first that $x_1 \neq x_2$. Since T is Hausdorff, there are disjoint open subsets V_1 and V_2 such that $x_1 \in V_1$ and $x_2 \in V_2$. Then $\widehat{V_1} = \widehat{T}_U \cap s^{-1}(V_1)$ and $\widehat{V_2} = \widehat{T}_U \cap s^{-1}(V_2)$ are disjoint and open in $\widehat{T}_U$, and $\boldsymbol{\gamma}(h_1, x_1) \in \widehat{V_1}$ and $\boldsymbol{\gamma}(h_2, x_2) \in \widehat{V_2}$.

Now assume that $x_1 = x_2 =: x$ but $h_1(x) \neq h_2(x)$. Take disjoint open subsets $W_1, W_2 \subset U$ such that $h_1(x) \in W_1$ and $h_2(x) \in W_2$. Then $\widehat{W_1} = \widehat{T}_U \cap t^{-1}(W_1)$ and $\widehat{W_2} = \widehat{T}_U \cap t^{-1}(W_2)$ are disjoint and open in $\widehat{T}_U$, and $\boldsymbol{\gamma}(h_1, x) \in \widehat{W_1}$ and $\boldsymbol{\gamma}(h_2, x) \in \widehat{W_2}$.

Finally, suppose that $x_1 = x_2 =: x$ and $h_1(x) = h_2(x) =: y$. Then $x \in T_{i_k} \cap U$ and $y \in T_{i_l} \cap U$ for some indexes k and l. Take open neighborhoods, V of x and W of y, such that $\overline{V} \subset T_{i_k} \cap U$, $\overline{W} \subset T_{i_l} \cap U$ and $h_1(\overline{V}) \cup h_2(\overline{V}) \subset W$. Define $\overline{S}_0$ and $\overline{S}_1$ by using V and W like in (4.1) and (4.2), and take an open subset $V_0 \subset T$ such that $x \in V_0$ and $\overline{V_0} \subset V$, as above. We can assume that $h_1, h_1 \in \overline{S}_1$. Then

$$\overline{\boldsymbol{\gamma}}(\mathcal{R}(h_1), x) = \boldsymbol{\gamma}(h_1, x_1) \neq \boldsymbol{\gamma}(h_2, x_2) = \overline{\boldsymbol{\gamma}}(\mathcal{R}(h_2), x) \; ,$$

and therefore $\mathcal{R}(h_1) \neq \mathcal{R}(h_2)$ in $\mathcal{R}(\overline{S}_1)$ by Lemma 4.3.8. Since $\mathcal{R}(\overline{S}_1)$ is Hausdorff (because it is a subspace of $C_{\text{c-o}}(\overline{V}, \overline{W})$), it follows that there are disjoint open subsets $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{R}(\overline{S}_1)$ such that $\mathcal{R}(h_1) \in \mathcal{N}_1$ and $\mathcal{R}(h_2) \in \mathcal{N}_2$. So $\mathcal{R}^{-1}(\mathcal{N}_1)$ and $\mathcal{R}^{-1}(\mathcal{N}_2)$ are disjoint open subsets of $\overline{S}_1$ with $h_1 \in \mathcal{R}^{-1}(\mathcal{N}_1)$ and $h_1 \in \mathcal{R}^{-1}(\mathcal{N}_1)$. Hence $\mathcal{M}_1 = \mathcal{R}^{-1}(\mathcal{N}_1) \cap \overline{S}_0$ and $\mathcal{M}_2 = \mathcal{R}^{-1}(\mathcal{N}_2) \cap \overline{S}_0$ are disjoint and open in $\overline{S}_0$, and therefore they are open in $\overline{S}$. Moreover $\mathcal{M}_1 \times V_0$ and $\mathcal{M}_2 \times V_0$ are saturated by the fibers of $\boldsymbol{\gamma} : \overline{S}_0 \times V_0 \to \boldsymbol{\gamma}(\overline{S}_0 \times V_0)$; in fact, if $(f, z) \in \overline{S}_0 \times V_0$ satisfies $\boldsymbol{\gamma}(f, z) = \boldsymbol{\gamma}(f', z)$ for some $f' \in \mathcal{M}_a$ ($a \in \{1, 2\}$), then

$$\overline{\boldsymbol{\gamma}}(\mathcal{R}(f), z) = \boldsymbol{\gamma}(f, z) = \boldsymbol{\gamma}(f', z) = \overline{\boldsymbol{\gamma}}(\mathcal{R}(f'), z) \; ,$$

giving $\mathcal{R}(f) = \mathcal{R}(f') \in \mathcal{N}_a$ by Lemma 4.3.8, and therefore $f \in \mathcal{R}^{-1}(\mathcal{N}_a) \cap \overline{S}_0 = \mathcal{M}_a$. It follows that $\boldsymbol{\gamma}(\mathcal{M}_1 \times V_0)$ and $\boldsymbol{\gamma}(\mathcal{M}_2 \times V_0)$ are open in $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ since $\boldsymbol{\gamma} : \overline{S}_0 \times V_0 \to \boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ is an identification because $\overline{S}_0 \times V_0$ is open in $\overline{S}_{\text{c-o}} * T$ and saturated by the fibers of $\boldsymbol{\gamma} : \overline{S}_{\text{c-o}} * T \to \widehat{T}$. Furthermore

$$\begin{aligned} \boldsymbol{\gamma}(\mathcal{M}_1 \times V_0) \cap \boldsymbol{\gamma}(\mathcal{M}_2 \times V_0) &= \overline{\boldsymbol{\gamma}}(\mathcal{N}_1 \times V_0) \cap \overline{\boldsymbol{\gamma}}(\mathcal{N}_2 \times V_0) \\ &= \overline{\boldsymbol{\gamma}}((\mathcal{N}_1 \cap \mathcal{N}_1) \times V_0) = \emptyset \end{aligned}$$

by the commutativity of the diagram (4.3), and $\boldsymbol{\gamma}(h_1, x) \in \boldsymbol{\gamma}(\mathcal{M}_1 \times V_0)$ and $\boldsymbol{\gamma}(h_2, x) \in \boldsymbol{\gamma}(\mathcal{M}_2 \times V_0)$. $\square$

Corollary 4.3.12. *$\overline{\boldsymbol{\gamma}} : \mathcal{R}(\overline{S}_1) \times \overline{V_0} \to \overline{\boldsymbol{\gamma}}(\mathcal{R}(\overline{S}_1) \times \overline{V_0})$ is a homeomorphism.*

Lemma 4.3.13. *$\widehat{T}_U$ is second countable.*

Proof. $\widehat{T}_U$ can be covered by a countable collection of open subsets of the type $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ as above. But $\boldsymbol{\gamma}(\overline{S}_0 \times V_0)$ is second countable because it is a subspace of $\boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0}) = \boldsymbol{\gamma}(\mathcal{R}(\overline{S}_1) \times \overline{V_0})$, which is homeomorphic to $\mathcal{R}(S_1) \times \overline{V_0}$, and this space is second countable because it is a subspace of the second countable space $C(\overline{V_0}, \overline{W_0}) \times \overline{V_0}$. □

Corollary 4.3.14. *$\widehat{T}_U$ is Polish.*

Proof. This follows from Corollary 4.3.7, Lemmas 4.3.11 and 4.3.13, and [29, Theorem 5.3]. □

Proposition 4.3.15. *$\widehat{T}$ is Polish and locally compact.*

Proof. First, let us prove that $\widehat{T}$ is Hausdorff. Take different points $\boldsymbol{\gamma}(g,x)$ and $\boldsymbol{\gamma}(g',x')$ in $\widehat{T}$. Let O, O', P and P' be relatively compact open neighborhoods of x, x', $g(x)$ and $g(x')$, respectively. Then

$$U_1 = U \cup O \cup O' \cup P \cup P'$$

is a relatively compact open subset of T that meets all $\mathcal{H}$-orbits. By Lemma 4.3.11, $\widehat{T}_{U_1}$ is a Hausdorff open subset of $\widehat{T}$ that contains $\boldsymbol{\gamma}(g,x)$ and $\boldsymbol{\gamma}(g',x')$. Hence $\boldsymbol{\gamma}(g,x)$ and $\boldsymbol{\gamma}(g',x')$ can be separated in $\widehat{T}_{U_1}$ by disjoint open neighborhoods in $\widehat{T}_{U_1}$, and therefore also in $\widehat{T}$.

Second, let us show that $\widehat{T}$ is locally compact. For $\boldsymbol{\gamma}(g,x) \in \widehat{T}$, let O and P be relatively compact open neighborhoods of x and $g(x)$, respectively. Then $U_1 = U \cup O \cup P$ is a relatively compact open set of T that meets all $\mathcal{H}$-orbits. By Corollary 4.3.7, it follows that $\widehat{T}_{U_1}$ is a locally compact open neighborhood of $\boldsymbol{\gamma}(g,x)$ in $\widehat{T}$. Hence $\boldsymbol{\gamma}(g,x)$ has a compact neighborhood in $\widehat{T}_{U_1}$, and therefore also in $\widehat{T}$.

Finally, let us show that $\widehat{T}$ is second countable. Since T is second countable (it is Polish) and locally compact, it can be covered by countably many relatively compact open subsets $O_n \subset T$. Then each $U_{n,m} = O_n \cup O_m \cup U$ is a relativelly compact open set of T that meets all $\mathcal{H}$-orbits. Hence, by Lemma 4.3.13, the sets $\widehat{T}_{U_{n,m}}$ are second countable and open in $\widehat{T}$. Moreover these sets form a countable cover of $\widehat{T}$ because, for any $\boldsymbol{\gamma}(g,x) \in \widehat{T}$, we have $x \in O_n$ and $g(x) \in O_m$ for some n and m, obtaining $\boldsymbol{\gamma}(g,x) \in \widehat{T}_{U_{n,m}}$. So $\widehat{T}$ is second countable.

Now the result follows by [29, Theorem 5.3]. □

Proposition 4.3.16. *$\hat{\pi} : \widehat{T} \to T \times T$ is proper.*

Proof. Let K be any compact subset of $T \times T$. Take any relatively compact open subset $U' \subset T$ meeting all $\mathcal{H}$-orbits such that $K \subset U' \times U'$. By applying Corollary 4.3.9 to any U', we get that $\hat{\pi}^{-1}(K)$ is compact in $\widehat{T}_{U'}$, and therefore in $\widehat{T}$. □

4.4 The space $\widehat{T}_0$

From now on, assume that the pseudogroup $\mathcal{H}$ is minimal, and therefore $\overline{\mathcal{H}}$ has only one orbit, which is T. Fix a point $x_0 \in U$, and let

$$\widehat{T}_0 = t^{-1}(x_0) = \{\, \boldsymbol{\gamma}(g,x) \mid g(x) = x_0 \,\} ,$$
$$\widehat{T}_{0,U} = \widehat{T}_0 \cap \widehat{T}_U .$$

Observe that $\widehat{T}_0$ is closed in $\widehat{T}$ and $\widehat{T}_{0,U}$ is open in $\widehat{T}_0$. Moreover $\hat{\pi}(\widehat{T}_0) = T$ and $\hat{\pi}(\widehat{T}_{0,U}) = U$ because T is the unique $\overline{\mathcal{H}}$-orbit; indeed, $\hat{\pi}(\boldsymbol{\gamma}(h,x)) = x$ for each $x \in T$ and any $h \in \overline{S}$ with $x \in \operatorname{dom} h$ and $h(x) = x_0$. Let $\hat{\pi}_0 := s : \widehat{T}_0 \to T$, which is continuous and surjective.

The following two corollaries are direct consequences of Proposition 4.3.15 (see [29, Theorem 3.11]) and Corollary 4.3.10.

Corollary 4.4.1. *$\widehat{T}_0$ is Polish and locally compact.*

Corollary 4.4.2. *The closure of $\widehat{T}_{0,U}$ in $\widehat{T}_0$ is compact.*

The following corollary is a direct consequence of Proposition 4.3.16 because $\hat{\pi}_0 : \widehat{T}_0 \to T$ can be identified with the restriction $\hat{\pi} : \widehat{T}_0 \to T \times \{x_0\} \equiv T$.

Corollary 4.4.3. *$\hat{\pi}_0 : \widehat{T}_0 \to T$ is proper.*

Proposition 4.4.4. *The fibers of $\hat{\pi}_0 : \widehat{T}_0 \to T$ are homeomorphic to each other.*

Proof. Let $x \in T$. Since T is the unique orbit of $\overline{\mathcal{H}}$, there is some $f \in \overline{S}$ with $f(x) = x_0$. Then the mapping $\boldsymbol{\gamma}(g,x) \mapsto \boldsymbol{\gamma}(gf^{-1}, x_0)$ defines a homeomorphism $\hat{\pi}_0^{-1}(x) \to \hat{\pi}_0^{-1}(x_0)$ whose inverse is given by $\boldsymbol{\gamma}(g_0, x_0) \mapsto \boldsymbol{\gamma}(g_0 f, x)$. □

Remark 32. In the case where $\hat{\pi}_0$ has local sections, the above argument would show that $\hat{\pi}_0$ is a fiber bundle.

4.5 The pseudogroup $\widehat{\mathcal{H}}_0$

For $h \in S$, define

$$\hat{h} : \hat{\pi}_0^{-1}(\operatorname{dom} h) \to \hat{\pi}_0^{-1}(\operatorname{im} h) , \quad \hat{h}(\boldsymbol{\gamma}(g,x)) = \boldsymbol{\gamma}(gh^{-1}, h(x)) ,$$

for $g \in S$, $x \in \operatorname{dom} g \cap \operatorname{dom} h$ with $g(x) = x_0$. Observe that $\hat{h}$ is well defined with image in $\widehat{T}_0$ because

$$t(\hat{h}(\boldsymbol{\gamma}(g,x))) = t(\boldsymbol{\gamma}(gh^{-1}, h(x))) = gh^{-1}h(x) = g(x) = x_0 .$$

The following result is elementary since $\hat{\pi}_0$ is surjective.

Lemma 4.5.1. *For any $h \in S$, we have $\hat\pi_0(\operatorname{dom}\hat h) = \operatorname{dom} h$ and $\hat\pi_0(\operatorname{im}\hat h) = \operatorname{im} h$, and the diagram*

$$\begin{array}{ccc} \operatorname{dom}\hat h & \xrightarrow{\hat h} & \operatorname{im}\hat h \\ {\scriptstyle\hat\pi_0}\downarrow & & \downarrow{\scriptstyle\hat\pi_0} \\ \operatorname{dom} h & \xrightarrow{h} & \operatorname{im} h \end{array}$$

is commutative.

Lemma 4.5.2. *If $O \subset T$ is open with $\operatorname{id}_O \in S$, then $\widehat{\operatorname{id}_O} = \operatorname{id}_{\hat\pi_0^{-1}(O)}$.*

Proof. For $g \in \overline{S}$ and $x \in \operatorname{dom} g \cap O$ with $g(x) = x_0$, we have

$$\widehat{\operatorname{id}_O}(\boldsymbol{\gamma}(g,x)) = \boldsymbol{\gamma}(g\operatorname{id}_O^{-1}, \operatorname{id}_O(x))) = \boldsymbol{\gamma}(g,x)\ . \quad \square$$

Lemma 4.5.3. *For $h, h' \in S$, we have $\widehat{hh'} = \hat h\widehat{h'}$.*

Proof. We have

$$\begin{aligned} \operatorname{dom}(\hat h\widehat{h'}) &= \widehat{h'}^{-1}(\operatorname{dom}\widehat{h'} \cap \operatorname{dom}\hat h) \\ &= \widehat{h'}^{-1}(\hat\pi_0^{-1}(\operatorname{dom} h' \cap \operatorname{dom} h)) \\ &= \{\, \boldsymbol{\gamma}(g,x) \mid g \in \overline{S},\ x \in \operatorname{dom} g,\ \boldsymbol{\gamma}(g,x) \in \operatorname{dom}\hat h, \\ &\qquad \hat h(\boldsymbol{\gamma}(g,x)) \in s^{-1}(\operatorname{dom} h' \cap \operatorname{dom} h)\,\} \\ &= \{\, \boldsymbol{\gamma}(g,x) \mid g \in \overline{S},\ x \in \operatorname{dom} g \cap \operatorname{dom} h, \\ &\qquad h(x) \in \operatorname{dom} h',\ g(x) = x_0\,\} \\ &= \hat\pi_0^{-1}(h^{-1}(\operatorname{dom} h' \cap \operatorname{dom} h)) \\ &= \hat\pi_0^{-1}(\operatorname{dom}(h'h)) \\ &= \operatorname{dom}\widehat{h'h}\ . \end{aligned}$$

Now let $\boldsymbol{\gamma}(g,x) \in \operatorname{dom}(\widehat{h'}\hat h) = \operatorname{dom}\widehat{h'h}$; thus $g \in \overline{S}$, $x \in \operatorname{dom} g \cap \operatorname{dom} h$, $h(x) \in \operatorname{dom} h'$ and $g(x) = x_0$. Then

$$\begin{aligned} \widehat{h'h}(\boldsymbol{\gamma}(g,x)) &= \boldsymbol{\gamma}(g(h'h)^{-1}, h'h(x)) \\ &= \boldsymbol{\gamma}(gh^{-1}(h')^{-1}, h'h(x)) \\ &= \widehat{h'}(\boldsymbol{\gamma}(gh^{-1}, h(x))) \\ &= \widehat{h'}\hat h(\boldsymbol{\gamma}(g,x))\ . \quad \square \end{aligned}$$

Corollary 4.5.4. *For $h \in S$, the map $\hat h$ is bijective with $\hat h^{-1} = \widehat{h^{-1}}$.*

Proof. By Lemmas 4.5.2 and 4.5.3, we have

$$\widehat{h^{-1}}\hat h = \widehat{h^{-1}h} = \widehat{\operatorname{id}_{\operatorname{dom} h}} = \operatorname{id}_{\hat\pi_0^{-1}(\operatorname{dom} h)} = \operatorname{id}_{\operatorname{dom}\hat h}\ . \quad \square$$

Lemma 4.5.5. *$\hat{h}$ is a homeomorphism for all $h \in S$.*

Proof. By Corollary 4.5.4, it is enough to prove that $\hat{h}$ is continuous, which holds because it can be expressed as the composition of the following continuous maps:

$$\begin{aligned} \hat{\pi}_0^{-1}(\operatorname{dom} h) &\xrightarrow{(\mathrm{id},\mathrm{const},h\hat{\pi}_0)} \hat{\pi}_0^{-1}(\operatorname{dom} h) \times \{h^{-1}\} \times \operatorname{im} h \\ &\xrightarrow{\mathrm{id}\times\boldsymbol{\gamma}} \hat{\pi}_0^{-1}(\operatorname{dom} h) \times \boldsymbol{\gamma}(\{h^{-1}\} \times \operatorname{im} h) \\ &\xrightarrow{\text{product}} \hat{\pi}_0^{-1}(\operatorname{im} h)\ , \end{aligned}$$

as can be checked on elements:

$$\begin{aligned} \boldsymbol{\gamma}(g,x) &\mapsto (\boldsymbol{\gamma}(g,x), h^{-1}, h(x)) \\ &\mapsto (\boldsymbol{\gamma}(g,x), \boldsymbol{\gamma}(h^{-1}, h(x))) \\ &\mapsto \boldsymbol{\gamma}(gh^{-1}, h(x)) = \hat{h}(\boldsymbol{\gamma}(g,x))\ . \quad \square \end{aligned}$$

Set $\widehat{S}_0 = \{\, \hat{h} \mid h \in S \,\}$, and let $\widehat{\mathcal{H}}_0$ be the pseudogroup on $\widehat{T}_0$ generated by $\widehat{S}_0$. Lemmas 4.5.3 and 4.5.5, and Corollary 4.5.4 give the following.

Corollary 4.5.6. *$\widehat{S}_0$ is a pseudo∗group on $\widehat{T}_0$.*

Lemma 4.5.7. *$\widehat{T}_{0,U}$ meets all orbits of $\widehat{\mathcal{H}}_0$.*

Proof. Let $\boldsymbol{\gamma}(g,x) \in \widehat{T}_0$ with $g \in \overline{S}$; thus $x \in \operatorname{dom} g$ and $g(x) = x_0$. Since U meets all orbits of $\mathcal{H}$, there is some $h \in S$ such that $x \in \operatorname{dom} h$ and $h(x) \in U$. Then $\boldsymbol{\gamma}(g,x) \in \operatorname{dom} \hat{h}$ and $\hat{h}(\boldsymbol{\gamma}(g,x)) = \boldsymbol{\gamma}(gh^{-1}, h(x))$ satisfies

$$\hat{\pi}_0(\hat{h}(\boldsymbol{\gamma}(g,x))) = \hat{\pi}_0(\boldsymbol{\gamma}(gh^{-1}, h(x))) = h(x) \in U\ .$$

Hence $\hat{h}(\boldsymbol{\gamma}(g,x)) \in \widehat{T}_{0,U}$ as desired. $\square$

Lemma 4.5.8. *The map $S_{\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \to \hat{h}$, is a homeomorphism.*

Proof. Suppose that $\widehat{h_1} = \widehat{h_2}$ for some $h_1,\ h_2 \in S$. Then $h_1 = h_2$ by Lemma 4.5.1. So the stated map is injective, and therefore it is bijective by the definition of $\widehat{S}_0$.

Take a subbasic open set of $S_{\text{c-o}}$, which is of the form $S \cap \mathcal{N}(K,O)$ for some compact K and open O in T. The set $\hat{\pi}_0^{-1}(K)$ is compact by Corollary 4.4.3 and $\hat{\pi}_0^{-1}(O)$ is open. Then the map of the statement is open because

$$\{\, \hat{h} \mid h \in \mathcal{N}(K,O) \cap S \,\} = \widehat{\mathcal{N}}(\hat{\pi}_0^{-1}(K), \hat{\pi}_0^{-1}(O)) \cap \widehat{S}_0$$

by Lemma 4.5.1, which is open in $\widehat{S}_{0,\text{c-o}}$.

To prove its continuity, let us first show that its restriction to $S_U = S \cap \mathcal{H}|_U$ is continuous. Fix $h_0 \in S_U$, and take relatively compact open subsets

$$V, V_0, W, V', V_0', W' \subset U \ ,$$

and indices k and k' such that:

$$\overline{V_0} \subset V \ , \quad \overline{V} \subset T_{i_k} \cap \operatorname{dom} h_0 \ , \tag{4.4}$$

$$\overline{V_0'} \subset V' \ , \quad \overline{V'} \subset T_{i_{k'}} \cap \operatorname{dom} h_0, \tag{4.5}$$

$$\overline{W} \subset W' \ , \quad \overline{W'} \subset T_{i_{k_0}} \ , \tag{4.6}$$

$$h_0^{-1}(\overline{V'}) \subset V \ , \tag{4.7}$$

$$h_0^{-1}(\overline{V_0}) \subset V \ . \tag{4.8}$$

Let $\overline{S}_0$ and $\overline{S}_1$ (respectively, $\overline{S}_0'$ and $\overline{S}_1'$) be defined like in (4.1) and (4.2), by using V and W (respectively, V' and W'). Then $\widehat{K} = \boldsymbol{\gamma}(\overline{S}_1 \times \overline{V_0})$ is compact in $\widehat{T}$ by Lemma 4.3.5, and $\widehat{O} = \boldsymbol{\gamma}(\overline{S}_0' \times V')$ is open in $\widehat{T}$ by Lemma 4.3.6 and Remark 31. Then $\widehat{K}_0 = \widehat{K} \cap \widehat{T}_0$ is compact and $\widehat{O}_0 = \widehat{O} \cap \widehat{T}_0$ is open in $\widehat{T}_0$. So $\widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0) \cap \widehat{S}_0$ is a subbasic open set of $\widehat{S}_{0,\text{c-o}}$.

Claim 2. $\hat{h}_0 \in \widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0)$.

Let $\boldsymbol{\gamma}(g, x) \in \widehat{K}_0$; thus $g \in \overline{S}_1$, $x \in \overline{V}_0 \cap \operatorname{dom} g$ and $g(x) = x_0$. The condition $g \in \overline{S}_1$ means that $g \in \overline{S}$, $\overline{V} \subset \operatorname{dom} g$ and $g(\overline{V}) \subset \overline{W}$. By (4.5) and (4.6), it follows that $\overline{V'} \subset \operatorname{dom} g h_0^{-1}$ and

$$g h_0^{-1}(\overline{V'}) \subset g(\overline{V}) \subset \overline{W} \subset W' \ .$$

Hence $g h_0^{-1} \in \overline{S}_0'$, obtaining that

$$\widehat{h_0}(\boldsymbol{\gamma}(g, x)) = \boldsymbol{\gamma}(g h_0^{-1}, h_0(x)) \in \widehat{O} \ ,$$

which completes the proof of Claim 2.

Claim 3. The sets $\widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0) \cap \widehat{S}_0$, constructed as above, form a local subbasis of $\widehat{S}_{0,\text{c-o}}$ at $\widehat{h_0}$.

This assertion follows because the sets of the type $\widehat{O}_0$ form a basis of the topology of $\operatorname{im} \widehat{h_0}$, and any compact subset of $\operatorname{dom} \widehat{h_0}$ is contained in a finite union of sets of the type of $\widehat{K}_0$.

The sets

$$\mathcal{N} = \mathcal{N}(\overline{V_0}, V') \cap \mathcal{N}(\overline{V'}, V)^{-1} \cap S_U$$

are open neigborhoods of h_0 by (4.7), (4.8), Proposition 3.1.11 and Corollary 4.2.2.

Claim 4. $\hat{h} \in \widehat{\mathcal{N}}(\widehat{K}_0, \widehat{O}_0)$ for all $h \in \mathcal{N}$.

Given $h \in \mathcal{N}$, we have $\overline{V'} \subset \operatorname{im} h$ and $h^{-1}(\overline{V'}) \subset V$. Let $\boldsymbol{\gamma}(g,x) \in \widehat{K}_0$; thus $x \in \overline{V_0} \cap \operatorname{dom} g$, $g(x) = x_0$, and we can assume that $g \in \overline{S}_1$, which means that $g \in \overline{S}$, $\overline{V} \subset \operatorname{dom} g$ and $g(\overline{V}) \subset \overline{W}$. Then $\overline{V'} \subset \operatorname{dom}(gh^{-1})$, $gh^{-1}(\overline{V'}) \subset \overline{W} \subset W'$ and $h(x) \in h(\overline{V_0}) \subset V'$. Therefore

$$\hat{h}(\boldsymbol{\gamma}(g,x)) = \boldsymbol{\gamma}(gh^{-1}, h(x)) \in \boldsymbol{\gamma}(\overline{S}'_0 \times V') \cap \widehat{T}_0 = \widehat{O}_0,$$

showing Claim 4.

Claims 3 and 4 show that the map $S_{U,\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \mapsto \hat{h}$, is continuous at h_0.

Now, let us prove that the whole map $S_{\text{c-o}} \to \widehat{S}_{0,\text{c-o}}$, $h \mapsto \hat{h}$, is continuous. Since the sets $\mathcal{N}(\widehat{K}, \widehat{O}) \cap \widehat{S}_0$, for small enough compact subsets $\widehat{K} \subset \widehat{T}_0$ and small enough open subsets $\widehat{O} \subset \widehat{T}_0$, form a subbasis of $\widehat{S}_{0,\text{c-o}}$, it is enough to prove that the inverse image of these subbasic sets are open in $S_{\text{c-o}}$. If $\widehat{K}$ and $\widehat{O}$ are small enough, we can assume that $\hat{\pi}_0(\widehat{K}) \subset \operatorname{dom} f_1$ and $\hat{\pi}_0(\widehat{O}) \subset \operatorname{dom} f_2$ for some $f_1, f_2 \in S$ with $\operatorname{im} f_1 \cup \operatorname{im} f_2 \subset U$.

By Proposition 3.5.4, if $\operatorname{dom} f_1$ is small enough, we have $\operatorname{im} h \subset \operatorname{dom} f_2$ for all $h \in S$ satisfying $\hat{\pi}_0(\widehat{K}) \subset \operatorname{dom} h \subset \operatorname{dom} f_1$ and $h\hat{\pi}_0(\widehat{K}) \subset \hat{\pi}_0(\widehat{O})$; i.e.,

$$S \cap \operatorname{Paro}(\operatorname{dom} f_1, T) \cap \mathcal{N}(\hat{\pi}_0(\widehat{K}), \hat{\pi}_0(\widehat{O})) \subset \operatorname{Paro}(\operatorname{dom} f_1, \operatorname{dom} f_2)\ .$$

Then, by Proposition 3.1.3, it is enough to prove that the inverse image of

$$\mathcal{N}(\widehat{K}, \widehat{O}) \cap \widehat{S}_0 \cap \operatorname{Paro}(\operatorname{dom} \widehat{f}_1, \operatorname{dom} \widehat{f}_2)$$

by the map

$$S \cap \operatorname{Paro}(\operatorname{dom} f_1, \operatorname{dom} f_2) \to \widehat{S}_0 \cap \operatorname{Paro}(\operatorname{dom} \widehat{f}_1, \operatorname{dom} \widehat{f}_2)\ , \quad h \mapsto \hat{h}\ ,$$

is open. But this follows from the previous case because the diagram

$$\begin{array}{ccc}
\widehat{S}_0 \cap \operatorname{Paro}(\operatorname{dom} \widehat{f}_1, \operatorname{dom} \widehat{f}_2) & \longrightarrow & \widehat{S}_0 \cap \operatorname{Paro}(\operatorname{im} \widehat{f}_1, \operatorname{im} \widehat{f}_2) \\
\uparrow & & \uparrow \\
S \cap \operatorname{Paro}(\operatorname{dom} f_1, \operatorname{dom} f_2) & \longrightarrow & S \cap \operatorname{Paro}(\operatorname{im} f_1, \operatorname{im} f_2)
\end{array}$$

is commutative by Lemma 4.5.3 and Corollary 4.5.4, where the vertical maps are given by $h \mapsto \hat{h}$ and the horizontal ones are defined by $\hat{h} \mapsto \widehat{f}_2^{\,-1} \hat{h} \widehat{f}_1$ and $h \mapsto f_2^{-1} h f_1$. $\square$

Since the compact generation of $\mathcal{H}$ is satisfied with the relatively compact open set U, there is a symmetric finite set $\{f_1, \dots, f_m\}$ generating $\mathcal{H}|_U$, which can be chosen in S, such that each f_a has an extension $\tilde{f}_a$ with $\overline{\operatorname{dom} f_a} \subset$

$\operatorname{dom} \tilde{f}_a$. We can also assume that $\tilde{f}_a$ is in S. Let $\widehat{\mathcal{H}}_{0,U} = \widehat{\mathcal{H}}|_{\widehat{T}_{0,U}}$. Obviously, each $\widehat{\tilde{f}}_a$ is an extenstion of $\widehat{f}_a$. Moreover

$$\begin{aligned}
\overline{\operatorname{dom} \widehat{f}_a} &= \overline{\widehat{T}_0 \cap \hat{\pi}_0^{-1}(\operatorname{dom} f_a)} \\
&\subset \widehat{T}_0 \cap \hat{\pi}_0^{-1}(\overline{\operatorname{dom} f_a}) \\
&\subset \widehat{T}_0 \cap \hat{\pi}_0^{-1}(\operatorname{dom} \tilde{f}_a) \\
&= \operatorname{dom} \widehat{\tilde{f}}_a \;.
\end{aligned}$$

Lemma 4.5.9. *The maps $\widehat{f}_a$ $(a \in \{1, \dots, m\})$ generate $\widehat{\mathcal{H}}_{0,U}$.*

Proof. $\widehat{\mathcal{H}}_{0,U}$ is generated by the maps of the form $\hat{h}$ with $h \in S_U$, and any such $\hat{h}$ can be written as a composition of maps $\widehat{f}_a$ around any $\boldsymbol{\gamma}(g, x) \in \operatorname{dom} \hat{h} = \hat{\pi}_0^{-1}(\operatorname{dom} h)$ by Lemma 4.5.3. □

Corollary 4.5.10. *$\widehat{\mathcal{H}}_0$ is compactly generated.*

Proof. We saw that $\widehat{T}_{0,U}$ is relatively compact in $\widehat{T}_0$ and meets all $\widehat{\mathcal{H}}_0$-orbits (Lemma 4.5.7), the maps $\widehat{f}_a$ generate $\widehat{\mathcal{H}}_{0,U}$ (Lemma 4.5.9), and each $\widehat{\tilde{f}}_a$ is an extension of each $\widehat{f}_a$ with $\overline{\operatorname{dom} \widehat{f}_a} \subset \operatorname{dom} \widehat{\tilde{f}}_a$. □

Recall that the sets T_{i_k} form a finite covering of $\overline{U}$ by open sets of T. Fix some index k_0 such that $x_0 \in T_{i_{k_0}}$. Let $\{W_k\}$ be a shrinking of $\{T_{i_k}\}$ as cover of $\overline{U}$ by open subsets of T; i.e., $\{W_k\}$ is a cover of $\overline{U}$ by open subsets of T with the same index set and $\overline{W_k} \subset T_{i_k}$ for all k. By applying Proposition 3.5.4 several times, we get finite covers, $\{V_a\}$ and $\{V'_u\}$, of $\overline{U}$ by open subsets of T, and shrinkings, $\{W_{0,k}\}$ of $\{W_k\}$ and $\{V_{0,a}\}$ of $\{V_a\}$, as covers of $\overline{U}$ by open subsets of T, such that the following properties hold:

- For all $h \in \mathcal{H}$ and $x \in \operatorname{dom} h \cap U \cap V_a \cap W_{0,k}$ with $h(x) \in U \cap W_{0,l}$, there is some $\tilde{h} \in S$ such that
$$\overline{V_a} \subset \operatorname{dom} \tilde{h} \cap W_k \;, \quad \boldsymbol{\gamma}(\tilde{h}, x) = \boldsymbol{\gamma}(h, x) \;, \quad \tilde{h}(\overline{V_a}) \subset W_l \;.$$

- For all $h \in \mathcal{H}$ and $x \in \operatorname{dom} h \cap U \cap V'_u \cap V_{0,a}$ with $h(x) \in U \cap V_{0,b}$, there is some $\tilde{h} \in S$ such that
$$\overline{V'_u} \subset \operatorname{dom} \tilde{h} \cap V_a \;, \quad \boldsymbol{\gamma}(\tilde{h}, x) = \boldsymbol{\gamma}(h, x) \;, \quad \tilde{h}(\overline{V'_u}) \subset V_b \;.$$

By the definition of $\overline{\mathcal{H}}$ and $\overline{S}$, it follows that these properties also hold for all $h \in \overline{\mathcal{H}}$ with $\tilde{h} \in \overline{S}$. Let $\{V'_{0,u}\}$ be a shrinking of $\{V'_u\}$ as a cover of $\overline{U}$ by open subsets of T. We have $x_0 \in W_{0,k_0} \cap V_{0,a_0} \cap V'_{0,u_0}$ for some indices k_0,

a_0 and u_0. For each a, let $\overline{S}_{0,a}$, $\overline{S}_{1,a} \subset \overline{S}$ be defined like $\overline{S}_0$ and $\overline{S}_1$ in (4.1) and (4.2) by using V_a and W_{k_0} instead of V and W. Take an index u such that $\overline{V'_u} \subset V_a$. The sets $V_{0,a} \cap V'_{0,u}$, defined in this way, form a cover of $\overline{U}$, obtaining that the sets $\widehat{T}_{a,u} = \boldsymbol{\gamma}(\overline{S}_{0,a} \times (V_{0,a} \cap V'_{0,u}))$ form a cover of $\overline{\widehat{T}_U}$ by open subsets of $\widehat{T}$ (Lemma 4.3.6), and therefore the sets $\widehat{T}_{0,a,u} = \widehat{T}_{a,u} \cap \widehat{T}_0$ form a cover of $\overline{\widehat{T}_{0,U}}$ by open subsets of $\widehat{T}_0$. Let $\widehat{T}_{0,U,a,u} = \widehat{T}_{0,U} \cap \widehat{T}_{a,u}$. Like in Section 4.3, let $\overline{\boldsymbol{\gamma}}$ denote the germ map defined on $C(\overline{V_a}, \overline{W_{k_0}}) \times \overline{V_a}$, and let $\mathcal{R} : \overline{S}_{1,a} \to C(\overline{V_a}, \overline{W_{k_0}})$ be the restriction map $f \mapsto f|_{\overline{V_a}}$. Then

$$\overline{\boldsymbol{\gamma}} : \mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}} \to \overline{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}) \tag{4.9}$$

is a homeomorphism by Corollary 4.3.12. Since $\overline{V_a}$ is compact, the compact-open topology on $\mathcal{R}_a(\overline{S}_{1,a})$ equals the topology induced by the supremum metric d_a on $C(\overline{V_a}, \overline{W_{k_0}})$, defined with the metric $d_{i_{k_0}}$ on $T_{i_{k_0}}$. Take some index k such that $\overline{V_a} \subset W_k$. Then the topology of $\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}$ is induced by the metric $d_{a,u,k}$ given by

$$d_{a,u,k}((g, y), (g', y')) = d_{i_k}(y, y') + d_a(g, g')$$

(recall that $\overline{W_k} \subset T_{i_k}$). Let $\hat{d}_{a,u,k}$ be the metric on $\overline{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}})$ that corresponds to $d_{a,u,k}$ by the homeomorphism (4.9); it induces the topology of $\overline{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}})$. Recall from the proof of Lemma 4.3.5 (see (4.3)) that

$$\overline{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}) = \boldsymbol{\gamma}(\mathcal{R}_a(\overline{S}) \times \overline{V_{0,a} \cap V'_{0,u}}) \ ,$$

which is contained in $\widehat{T}$. Then the restriction $\hat{d}_{0,a,u,k}$ of $\hat{d}_{a,u,k}$ to

$$\overline{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}) \cap \widehat{T}_0$$

induces the topology of this space. Moreover, according to the proof of Corollary 4.3.7, we get

$$\widehat{T}_{a,u} \subset \overline{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}) \ ,$$

and therefore

$$\widehat{T}_{0,a,u} \subset \overline{\boldsymbol{\gamma}}(\mathcal{R}_a(\overline{S}_{1,a}) \times \overline{V_{0,a} \cap V'_{0,u}}) \cap \widehat{T}_0 \ .$$

For any index v, define $\overline{S}'_{0,v}$ and $\overline{S}'_{1,v}$ like $\overline{S}_{0,a}$ and $\overline{S}_{1,a}$, by using V'_v instead of V_a (also like $\overline{S}_0$ and $\overline{S}_1$ in (4.1) and (4.2) by using V'_v and W_{k_0} instead of V and W). Let $\mathcal{R}'_v : \overline{S}'_{1,v} \to C(\overline{V'_v}, \overline{W_{k_0}})$ denote the restriction map. Again, the compact-open topology on $\mathcal{R}'_v(\overline{S}'_{1,v})$ equals the topology induced by the supremun metric d'_v on $C(\overline{V'_v}, \overline{W_{k_0}})$, defined with the metric $d_{i_{k_0}}$ on $T_{i_{k_0}}$ (recall that $\overline{W}_{k_0} \subset T_{i_{k_0}}$). Take indices b and l such that $\overline{V'_v} \subset V_b$ and $\overline{V'_b} \subset W_l$. Then we can consider the restricion map

$$\mathcal{R}^v_b : C(\overline{V_b}, \overline{W_{k_0}}) \to C(\overline{V'_b}, \overline{W_{k_0}}) \ .$$

Its restricion $\mathcal{R}_b^v : \mathcal{R}_b(\overline{S}_{1,b}) \to \mathcal{R}'_v(\overline{S}'_{1,v})$ is injective by Remark 30, and surjective by Remark 29. So $\mathcal{R}_b^v : \mathcal{R}_b(\overline{S}_{1,b}) \to \mathcal{R}'_v(\overline{S}'_{1,v})$ is a continuous bijection between compact Hausdorff spaces, obtaining that it is a homeomorphism. Then, by compactness, it is a uniform homeomorphism with repect to the supremum metrics d_b and d'_v. Since b and v run in finite families of incices, there is a mapping $\varepsilon \to \delta_1(\varepsilon) > 0$, for $\varepsilon > 0$, such that

$$d'_v(\mathcal{R}_b^v\mathcal{R}_b(f), \mathcal{R}_b^v\mathcal{R}_b(f')) < \delta_1(\varepsilon) \Longrightarrow d_b(\mathcal{R}_b(f), \mathcal{R}_b(f')) < \varepsilon \tag{4.10}$$

for all $f, f' \in \overline{S}_{1,b}$, and indices v and b.

Lemma 4.5.11. *$\widehat{\mathcal{H}}_{0,U}$ satisfies the equicontinuity condition with $\widehat{S}_{0,U} = \widehat{S}_0 \cap \widehat{\mathcal{H}}_{0,U}$ and the quasi-local metric represented by the family $\{\widehat{T}_{0,U,a,u}, \hat{d}_{0,a,u,k}\}$.*

Proof. Let $h \in S$, and take

$$\boldsymbol{\gamma}(g, y), \boldsymbol{\gamma}(g', y') \in \widehat{T}_{0,U,a,u} \cap \hat{h}^{-1}(\widehat{T}_{0,U,b,v}) ,$$

where $g, g' \in \overline{S}_{0,a}$ and $y, y' \in V_{0,a} \cap V'_{0,u}$ with $g(y) = g(y') = x_0$. Take some indices k and l such that $\overline{V_a} \subset W_k$ and $\overline{V_b} \subset W_l$ (recall that $\overline{W_k} \subset T_{i_k}$ and $\overline{W_l} \subset T_{i_l}$). By Remark 29, we can assume that $\operatorname{dom} h = T_{i_k}$. Then

$$\hat{h}(\boldsymbol{\gamma}(g, y)) = \boldsymbol{\gamma}(gh^{-1}, h(y)) ,$$
$$\hat{h}(\boldsymbol{\gamma}(g', y')) = \boldsymbol{\gamma}(g'h^{-1}, h(y'))$$

belong to $\widehat{T}_{0,U,b,v}$, which means that $h(y), h(y') \in V_{0,b} \cap V'_{0,v}$ and there are $f, f' \in \overline{S}_{0,b}$ so that

$$\boldsymbol{\gamma}(f, h(y)) = \boldsymbol{\gamma}(gh^{-1}, h(y)) , \tag{4.11}$$
$$\boldsymbol{\gamma}(f', h(y')) = \boldsymbol{\gamma}(g'h^{-1}, h(y')) ; \tag{4.12}$$

in particular, $\overline{V_b} \subset \operatorname{dom} f \cap \operatorname{dom} f'$. In fact, we can assume that $\operatorname{dom} f = \operatorname{dom} f' = T_{i_l}$ by Remark 29. Observe that the image of h may not be included in T_{i_l}, and the images of f, f', g and g' may no be included in $T_{i_{k_0}}$.

Claim 5. $\overline{V'_v} \subset \operatorname{im} h$ and $h^{-1}(\overline{V'_v}) \subset V_a$.

By the assumptions on $\{V'_w\}$, since

$$h(y) \in U \cap V'_v \cap V_{0,b} \cap \operatorname{dom} h^{-1} , \quad h^{-1}(h(y)) = y \in U \cap V'_u \cap V_{0,a} ,$$

there is some $\widetilde{h^{-1}} \in S$ such that

$$\overline{V'_v} \subset \operatorname{dom} \widetilde{h^{-1}} \cap V_b , \quad \widetilde{h^{-1}}(\overline{V'_v}) \subset V_a ,$$
$$\boldsymbol{\gamma}(\widetilde{h^{-1}}, h(y)) = \boldsymbol{\gamma}(h^{-1}, h(y)) ;$$

indeed, we can suppose that $\operatorname{dom}\widetilde{h^{-1}} = T_{i_{k_0}}$ by Remark 29. Then

$$\widetilde{h^{-1}}(\overline{V'_v}) \subset V_a \subset T_{i_k} = \operatorname{dom} h\ ,$$

obtaining $\overline{V'_v} \subset \operatorname{dom}(h\widetilde{h^{-1}})$. Moreover

$$\boldsymbol{\gamma}(h\widetilde{h^{-1}}, h(y)) = \boldsymbol{\gamma}(\mathrm{id}_T, h(y))\ .$$

Therefore $h\widetilde{h^{-1}} = \mathrm{id}_{\operatorname{dom}(h\widetilde{h^{-1}})}$ because $h\widetilde{h^{-1}} \in S$ since $h, \widetilde{h^{-1}} \in S$. So $h\widetilde{h^{-1}} = \mathrm{id}_T$ on some neighborhood of $\overline{V'_v}$, and therefore $\overline{V'_v} \subset \operatorname{im} h$ and $h^{-1} = \widetilde{h^{-1}}$ on $\overline{V'_v}$. Thus $h^{-1}(\overline{V'_v}) = \widetilde{h^{-1}}(\overline{V'_v}) \subset V_a$, which shows Claim 5.

By Claim 5 and since $\overline{V_a} \subset \operatorname{dom} g \cap \operatorname{dom} g'$ because $g, g' \in \overline{S}_{0,a}$, we get

$$\overline{V'_v} \subset \operatorname{dom}(gh^{-1}) \cap \operatorname{dom}(g'h^{-1})\ . \tag{4.13}$$

Since $f, f' \in \overline{S}_{0,b}$, we have $\overline{V_b} \subset \operatorname{dom} f \cap \operatorname{dom} f'$ and $f(\overline{V_b}) \cup f'(\overline{V_b}) \subset W_{k_0}$. On the other hand, it follows from (4.11) and (4.12) that $fh(y) = f'h(y') = x_0$ and

$$\boldsymbol{\gamma}(gh^{-1}f^{-1}, x_0) = \boldsymbol{\gamma}(g'h^{-1}f'^{-1}, x_0) = \boldsymbol{\gamma}(\mathrm{id}_T, x_0)\ .$$

Moreover

$$f(\overline{V'_v}) \subset \operatorname{dom}(gh^{-1}f^{-1})\ , \quad f'(\overline{V'_v}) \subset \operatorname{dom}(g'h^{-1}f'^{-1})$$

by (4.13). So, by Remark 30, $gh^{-1}f^{-1} = \mathrm{id}_T$ in some neighborhood of $f(\overline{V'_v})$, and $g'h^{-1}f'^{-1} = \mathrm{id}_T$ in some neighborhood of $f'(\overline{V'_v})$. Thus $gh^{-1} = f$ and $g'h^{-1} = f'$ on some neighborhood of $\overline{V'_v}$; in particular,

$$\mathcal{R}^v_b\mathcal{R}_b(f) = gh^{-1}|_{\overline{V'_v}}\ , \quad \mathcal{R}^v_b\mathcal{R}_b(f') = g'h^{-1}|_{\overline{V'_v}}\ .$$

Consider the mappings $\varepsilon \mapsto \delta(\varepsilon) > 0$ and $\varepsilon \mapsto \delta_1(\varepsilon) > 0$ satisfying Remark 27 and (4.10). Then, for each $\varepsilon > 0$, define

$$\hat{\delta}(\varepsilon) = \min\{\delta(\varepsilon/2), \delta_1(\varepsilon/2)\}\ .$$

Given any $\varepsilon > 0$, suppose that

$$\hat{d}_{0,a,u,k}(\boldsymbol{\gamma}(g,y), \boldsymbol{\gamma}(g',y')) < \hat{\delta}(\varepsilon)\ .$$

This means that

$$d_{a,u,k}((\mathcal{R}_a(g), y), (\mathcal{R}_a(g'), y')) < \hat{\delta}(\varepsilon)\ ,$$

or, equivalently,

$$d_{i_k}(y, y') + \sup_{x \in \overline{V_a}} d_{i_{k_0}}(g(x), g(x')) < \hat{\delta}(\varepsilon)\ .$$

Therefore

$$d_{i_k}(y,y') < \delta(\varepsilon/2)\ , \tag{4.14}$$

$$\sup_{x\in\overline{V_a}} d_{i_{k_0}}(g(x),g(x')) < \delta_1(\varepsilon/2)\ . \tag{4.15}$$

From (4.14) and Remark 27, it follows that

$$d_{i_l}(h(y),h(y')) < \varepsilon/2 \tag{4.16}$$

since $h\in S\subset\overline{S}$ and $y,y'\in T_{i_k}\cap h^{-1}(T_{i_l}\cap\operatorname{im} h)$. On the other hand, by Claim 5 and (4.15), we get

$$\begin{aligned}
d'_v(\mathcal{R}^v_b\mathcal{R}_b(f),\mathcal{R}^v_b\mathcal{R}_b(f')) &= \sup_{z\in\overline{V'_v}} d_{i_{k_0}}(gh^{-1}(z),g'h^{-1}(z))\\
&= \sup_{x\in h^{-1}(\overline{V'_v})} d_{i_{k_0}}(g(x),g'(x))\\
&\le \sup_{x\in\overline{V_a}} d_{i_{k_0}}(g(x),g'(x))\\
&= d_a(\mathcal{R}_a(g),\mathcal{R}_a(g'))\\
&< \delta_1(\varepsilon/2)\ .
\end{aligned}$$

So, by (4.10),

$$d_b(\mathcal{R}_b(f),\mathcal{R}_b(f')) < \varepsilon/2\ . \tag{4.17}$$

From (4.16) and (4.17), we get

$$\begin{aligned}
&\hat{d}_{0,b,v,l}(\hat{h}(\boldsymbol{\gamma}(g,y)),\hat{h}(\boldsymbol{\gamma}(g',y')))\\
&\qquad= \hat{d}_{0,b,v,l}(\boldsymbol{\gamma}(f,h(y)),\boldsymbol{\gamma}(f',h(y')))\\
&\qquad= d_{b,v,l}((\mathcal{R}_b(f),h(y)),(\mathcal{R}_b(f'),h(y')))\\
&\qquad= d_{i_l}(h(y),h(y'))+d_b(\mathcal{R}_b(f),\mathcal{R}_b(f'))\\
&\qquad< \varepsilon\ . \quad \square
\end{aligned}$$

Corollary 4.5.12. *$\widehat{\mathcal{H}}_0$ is equicontinuous.*

Proof. $\widehat{\mathcal{H}}_0$ is equivalent to $\widehat{\mathcal{H}}_{0,U}$ by Lemma 4.5.7. Thus the result follows from Lemma 4.5.11 because equicontinuity is preserved by equivalences. $\square$

Lemma 4.5.13. *$\widehat{\mathcal{H}}_0$ is minimal.*

Proof. By Lemma 4.5.7, it is enough to prove that $\widehat{\mathcal{H}}_{0,U}$ is minimal. Let $\boldsymbol{\gamma}(g,y),\boldsymbol{\gamma}(g',y')\in\widehat{T}_{0,U}$ with $g,g'\in\overline{S}$, $y\in\operatorname{dom} g\cap U$, $y'\in\operatorname{dom} g'\cap U$ and $g(y)=g'(y')=x_0$. Take indices k and k' such that $y\in T_{i_k}$ and $y'\in T_{i_{k'}}$. We can assume that $\operatorname{dom} g=T_{i_k}$ and $\operatorname{dom} g'=T_{i_{k'}}$ by Remark 29.

We have $f = g^{-1}g' \in \overline{S}$, $y' \in \operatorname{dom}(g^{-1}g')$ and $g^{-1}g'(y') = y$. By Remark 29, there exists $\tilde{f} \in \overline{S}$ with $\operatorname{dom}\tilde{f} = T_{i_{k'}}$ and $\boldsymbol{\gamma}(\tilde{f}, y') = \boldsymbol{\gamma}(f, y')$. By the definition of $\overline{S}$, there is a sequence f_n in S with $\operatorname{dom} f_n = T_{i_{k'}}$ and $f_n \to f$ in $C_{\text{c-o}}(T_{i_k}, T)$ as $n \to \infty$; in particular, $f_n(y') \to f(y') = y$. So we can assume that $f_n(y') \in T_{i_k}$ for all n.

Take some relatively compact open neighborhood V of y' such that $\overline{V} \subset \operatorname{dom}(g\tilde{f}) \cap \operatorname{dom}(gf)$ and $\tilde{f} = f$ in some neighborhood of $\overline{V}$. Since $f_n \to \tilde{f}$ in $\overline{S}_{\text{c-o}}$ as $n \to \infty$, we get $gf_n \to g\tilde{f}$ and $f_n^{-1} \to \tilde{f}^{-1}$ by Propositions 3.1.11 and 4.2.1. So $\overline{V} \subset \operatorname{dom}(gf_n)$ and $y \in \operatorname{dom} f_n^{-1} = \operatorname{im} f_n$ for n large enough, and $f_n^{-1}(y) \to \tilde{f}^{-1}(y) = y'$. Moreover, $gf_n|_V \to g\tilde{f}|_V = gf|_V = g'|_V$ in $C_{\text{c-o}}(V, T)$. So $\boldsymbol{\gamma}(gf_n, f_n^{-1}(y)) \to \boldsymbol{\gamma}(g', y')$ in $\widehat{T}_{0,U}$ by Proposition 3.1.2 and the definition of the topology of $\widehat{T}$. Thus, with $h_n = f_n^{-1} \in S$, we get

$$\widehat{h_n}(\boldsymbol{\gamma}(g, y)) = \boldsymbol{\gamma}(gh_n^{-1}, h_n(y)) = \boldsymbol{\gamma}(gf_n, f_n^{-1}(y)) \to \boldsymbol{\gamma}(g', y')\ ,$$

obtaining that $\boldsymbol{\gamma}(g', y')$ is in the closure of $\widehat{\mathcal{H}}_{0,U}$-orbit of $\boldsymbol{\gamma}(g, y)$. $\square$

By Lemma 4.5.1, the map $\hat{\pi}_0 : \widehat{T}_0 \to T$ generates a morphisms of pseudogroups $\widehat{\Pi}_0 : \widehat{\mathcal{H}}_0 \to \mathcal{H}$.

Let $\widehat{\overline{\mathcal{H}}}_0$ be the pseudogroup on $\widehat{T}_0$ defined like $\widehat{\mathcal{H}}$ by taking the maps h in $\overline{S}$ instead of S; thus it is generated by $\widehat{\overline{S}}_0 = \{\, \hat{h} \mid h \in \overline{S}\,\}$. Observe that $\widehat{\overline{\mathcal{H}}}_0$, $\overline{S}$ and $\widehat{\overline{S}}_0$ satisfy the obvious versions of Lemmas 4.5.1–4.5.3, 4.5.5, 4.5.7 and 4.5.8, and Corollaries 4.5.4 and 4.5.6 (Section 4.5). In particular, $\widehat{\overline{S}}_0$ is a pseudo$*$group, and $\widehat{T}_{0,U}$ meets all orbits of $\widehat{\overline{\mathcal{H}}}_0$. The restriction of $\widehat{\overline{\mathcal{H}}}_0$ to $\widehat{T}_{0,U}$ will be denoted by $\widehat{\overline{\mathcal{H}}}_{0,U}$.

Lemma 4.5.14. $\overline{\widehat{\mathcal{H}}_0} = \widehat{\overline{\mathcal{H}}}_0$

Proof. By the version of Lemma 4.5.8 for $\overline{S}$ and $\widehat{\overline{S}}_0$, the set $\widehat{S}_0$ is dense in $\widehat{\overline{S}}_{0,\text{c-o}}$. Then the result follows easily by Proposition 3.1.2 and the definition of $\overline{\widehat{\mathcal{H}}_0}$ (see Theorem 3.5.6 and Remark 22). $\square$

Lemma 4.5.15. $\overline{\widehat{\mathcal{H}}}_0$ *is locally free and strongly quasi-analytic.*

Proof. Let $\hat{h} \in \widehat{\overline{S}}_0$ for $h \in \overline{S}$, and $\boldsymbol{\gamma}(g, x) \in \operatorname{dom}\hat{h}$ for $g \in \overline{S}$ and $x \in \operatorname{dom} g \cap \operatorname{dom} h$ with $g(x) = x_0$. Suppose that $\hat{h}(\boldsymbol{\gamma}(g, x)) = \boldsymbol{\gamma}(g, x)$. This means $\boldsymbol{\gamma}(gh^{-1}, h(x)) = \boldsymbol{\gamma}(g, x)$. So $h(x) = x$ and $gh^{-1} = g$ on some neighborhood of x, and therefore $h = \mathrm{id}_T$ on some neighborhood of x. It follows that $\hat{h} = \mathrm{id}_{\widehat{T}_0}$ on some neighborhood of $\boldsymbol{\gamma}(g, x)$ by Lemma 4.5.2. Moreover $h = \mathrm{id}_{\operatorname{dom} h}$ by the strong quasi-analiticity condition of $\overline{\mathcal{H}}$ since $h \in \overline{S}$. Hence $\hat{h} = \mathrm{id}_{\operatorname{dom}\hat{h}}$ by Lemma 4.5.2. $\square$

Proposition 4.5.16. *There is some locally compact, metrizable and separable local group G and some dense finitely generated sub-local group $\Gamma \subset G$ such that $\widehat{\mathcal{H}}_0$ is equivalent to the pseudogroup defined by the local action of Γ on G by local left translations.*

Proof. This follows from Remark 24 (see also Theorem 3.5.10) since $\widehat{\mathcal{H}}_0$ is compactly generated (Corollary 4.5.10) and equicontinuous (Corollary 4.5.12), and $\overline{\widehat{\mathcal{H}}_0}$ is strongly locally free (Lemma 4.5.15). □

Definition 4.5.17. In Proposition 4.5.16, it will be said that the local isomorphism class of G is the *structural local group* of $\mathcal{H}$.

This definition makes sense by Lemma 3.5.8 and the results of the next section. The following result is elementary.

Proposition 4.5.18. *Let G be a Polish locally compact local group, $\Gamma \subset G$ a dense local subgroup with a left invariant metric, and $\mathcal{H}$ the pseudogroup generated by the local action of Γ on G by local left translations (Example 3.5.7). Suppose that $\mathcal{H}$ is compactly generated. Then the structural local group of $\mathcal{H}$ is represented by G.*

4.6 Independence of the choices involved

First, let us prove that $\widehat{T}_0$ and $\widehat{\mathcal{H}}_0$ are independent of the choice of the point x_0 up to an equivalence generated by a homeomorphism. Let x_1 be another point of T, and let $\widehat{T}_1$, $\hat{\pi}_1$, $\widehat{S}_1$ and $\widehat{\mathcal{H}}_1$ be constructed like $\widehat{T}_0$, $\hat{\pi}_0$, $\widehat{S}_0$ and $\widehat{\mathcal{H}}_0$ by using x_1 instead of x_0. Now, for each $h \in S$, let us use the notation $\hat{h}_0 := \hat{h} \in \widehat{S}_0$, and let $\hat{h}_1 : \hat{\pi}_1^{-1}(\operatorname{dom} h) \to \hat{\pi}_1^{-1}(\operatorname{im} h)$ be the map in $\widehat{S}_1$ defined like $\hat{h}$ by using x_1 instead of x_0.

Proposition 4.6.1. *There is a homeomorphism $\theta : \widehat{T}_0 \to \widehat{T}_1$ that generates an equivalence $\Theta : \widehat{\mathcal{H}}_0 \to \widehat{\mathcal{H}}_1$ and so that $\hat{\pi}_0 = \hat{\pi}_1\theta$.*

Proof. Since the $\mathcal{H}$-orbits are dense, there is some $f_0 \in \overline{S}$ such that $x_0 \in \operatorname{dom} f_0$ and $f_0(x_0) = x_1$. Let $\theta : \widehat{T}_0 \to \widehat{T}_1$ be defined by $\theta(\boldsymbol{\gamma}(f,x)) = \boldsymbol{\gamma}(f_0 f, x)$. This θ is continuous because $\phi(\boldsymbol{\gamma}(f,x)) = \boldsymbol{\gamma}(f_0,x)\,\boldsymbol{\gamma}(f,x)$. So θ is a homeomorphism because f_0^{-1} defines θ^{-1} in the same way. We also have $\hat{\pi}_0 = \hat{\pi}_1\theta$ since θ preserves the source of each germ. For each $h \in S$, we have $\operatorname{dom} \hat{h}_1 = \theta(\operatorname{dom} \hat{h}_0)$ because $\hat{\pi}_0 = \hat{\pi}_1\theta$, and $\hat{h}_1\theta = \theta$ since

$$\begin{aligned}
\hat{h}_1\theta(\boldsymbol{\gamma}(f,x)) &= \hat{h}_1(\boldsymbol{\gamma}(f_0 f, x)) \\
&= \boldsymbol{\gamma}(f_0 f h^{-1}, h(x)) \\
&= \theta(\boldsymbol{\gamma}(f h^{-1}, h(x))) \\
&= \theta(\hat{h}_0(\boldsymbol{\gamma}(f,x)))
\end{aligned}$$

for all $\boldsymbol{\gamma}(f,x) \in \operatorname{dom} \hat{h}_0$. It follows easily that θ generates an étale morphism $\Theta : \widehat{\mathcal{H}}_0 \to \widehat{\mathcal{H}}_1$, which is an equivalence because θ^{-1} generates Θ^{-1}. $\square$

Now, let us show that the topology of $\widehat{T}$ is independent of the choice of S. Therefore the topology of $\widehat{T}_0$ will be independent of the choice of S as well. Let $S', S'' \subset \mathcal{H}$ be two sub-pseudo$*$groups generating $\mathcal{H}$ and satisfying the conditions of Section 4.1. With the notation of Section 4.2, we have to prove the following.

Proposition 4.6.2. $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} = \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$.

Proof. Observe that $S' \cap S''$ is a sub-pseudo$*$group of $\mathcal{H}$. It also generates $\mathcal{H}$ because S' and S'' are local. Moreover $S' \cap S''$ obviously satisfies all other properties required in Section 4.1; note that a refinement of $\{T_i\}$ may be necessary to get the properties stated in Remarks 27–30 with $S' \cap S''$. Hence the result follows from the special case where $S' \subset S''$. With this assumption, the identity map $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} \to \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$ is continuous because the diagram

$$\begin{tikzcd}
\overline{S'}_{\text{c-o}} \arrow[r, "\text{inclusion}"] \arrow[d, "\gamma"'] & \overline{S''}_{\text{c-o}} \arrow[d, "\gamma"] \\
\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} \arrow[r, "\text{identity}"] & \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}
\end{tikzcd}$$

is commutative, where the vertical maps are identifications and the top map is continuous.

For any compact subset $Q \subset T$, let $s^{-1}(Q)_{\overline{S'},\text{c-o}}$ and $s^{-1}(Q)_{\overline{S''},\text{c-o}}$ denote the spaces obtained by endowing $s^{-1}(Q)$ with the restriction of the topologies of $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}}$ and $\overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$, respectively. They are compact and Hausdorff by Corollary 4.3.9 and Proposition 4.3.15. So $s^{-1}(Q)_{\overline{S'},\text{c-o}} = s^{-1}(Q)_{\overline{S''},\text{c-o}}$ because the identity map $s^{-1}(Q)_{\overline{S'},\text{c-o}} \to s^{-1}(Q)_{\overline{S''},\text{c-o}}$ is continuous. Hence, for any $\boldsymbol{\gamma}(f,x) \in \overline{\mathfrak{G}}$ and a compact neighborhood Q of x in T, the set $s^{-1}(Q)$ is a neighborhood of $\boldsymbol{\gamma}(f,x)$ in $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}}$ and $\overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$ with $s^{-1}(Q)\ {}_{\mathit{overlineS'},\text{c-o}} = s^{-1}(Q)_{\overline{S''},\text{c-o}}$. This shows that the identity map $\overline{\mathfrak{G}}_{\overline{S'},\text{c-o}} \to \overline{\mathfrak{G}}_{\overline{S''},\text{c-o}}$ is a local homeomorphism, and therefore a homeomorphism. $\square$

Let T' be an open subset of T containing x_0, which meets all orbits because $\mathcal{H}$ is minimal. Then use T', $\mathcal{H}' = \mathcal{H}|_{T'}$ and $S' = S \cap \mathcal{H}'$ to define $\widehat{T'}_0$, $\hat{\pi}'_0$, $\widehat{S'}_0$ and $\widehat{\mathcal{H}'}_0$ like $\widehat{T}_0$, $\hat{\pi}_0$, $\widehat{S}_0$ and $\widehat{\mathcal{H}}_0$. The proof of the following result is elementary.

Proposition 4.6.3. *There is a canonical identity of topological spaces,* $\widehat{T'}_0 \equiv \hat{\pi}_0^{-1}(T')$*, so that* $\hat{\pi}'_0 \equiv \hat{\pi}_0|_{\widehat{T'}_0}$ *and* $\widehat{\mathcal{H}'}_0 = \widehat{\mathcal{H}}_0|_{\widehat{T'}_0}$.

Corollary 4.6.4. *Let* $\mathcal{H}$ *and* $\mathcal{H}'$ *be minimal equicontinuous compactly generated pseudogroups on locally compact Polish spaces such that* $\overline{\mathcal{H}}$ *and* $\overline{\mathcal{H}'}$ *are strongly quasi-analytic. If* $\mathcal{H}$ *is equivalent to* $\mathcal{H}'$*, then* $\widehat{\mathcal{H}}_0$ *is equivalent to* $\widehat{\mathcal{H}'}_0$.

Proof. This is a direct consequence of Propositions 4.6.1 and 4.6.3. □

CHAPTER 5

Molino's theory for equicontinuous foliated spaces

Contents

5.1 Molino's theory for equicontinuous foliated spaces

Let $(X, \mathcal{F})$ be a compact minimal foliated space that is equicontinuous and such that the closure of its holonomy pseudogroup is strongly quasi-analytic. Let $\{U_i, p_i, h_{ij}\}$ be a defining cocyle of $\mathcal{F}$ induced by a regular foliated atlas, where $p_i : U_i \to T_i$. Let $\mathcal{H}$ denote the corresponding representative of the holonomy pseudogroup on $T = \bigsqcup_i T_i$, which satisfies the conditions of Section 4.1

Fix and index i_0 and a point $x_0 \in U_{i_0}$. Let $\hat{\pi}_0 : \widehat{T}_0 \to T$ and $\widehat{\mathcal{H}}_0$ be defined like in Sections 4.4 and 4.5, by using T, $\mathcal{H}$, the point $p_{i_0}(x_0) \in T_{i_0} \subset T$, and a complete sub-pseudo$*$group $S \subset \mathcal{H}$ generating $\mathcal{H}$ and satisfying the conditions of Section 4.1. For instance, S can be the localization of the pseudo$*$group consisting of all compositions of maps h_{ij} if $\{U_i, p_i, h_{ij}\}$.

With the notation $\widehat{T}_{i,0} = \hat{\pi}_0^{-1}(T_i) \subset \widehat{T}_0$, let

$$\check{X}_0 = \bigsqcup_i U_i \times \widehat{T}_{i,0} = \bigcup_i U_i \times \{i\} \times \widehat{T}_{i,0} \ ,$$

endowed with the corresponding topological sum of the product topologies, and consider its closed subspace

$$\widetilde{X}_0 = \{ (x, i, \gamma) \in \check{X}_0 \mid p_i(x) = \hat{\pi}_0(\gamma) \} \subset \check{X}_0 \ .$$

Two elements (x, i, γ) and (y, j, δ) of $\widetilde{X}_0$ are declared to be equivalent if $x = y$ and

$$\gamma = \widehat{h_{ji}}(\delta) \ . \tag{5.1}$$

Since $h_{ij}p_i(x) = p_j(x)$, $h_{ji}^{-1} = h_{ij}$ and $h_{ik} = h_{jk}h_{ij}$, it follows that this is an equivalence relation on $\widetilde{X}_0$, denoted by "$\sim$", and the equivalence class of each triple (x, i, γ) is denoted by $[x, i, \gamma]$. Let $\widehat{X}_0$ be the corresponding quotient space and $q : \widetilde{X}_0 \to \widehat{X}_0$ the quotient map. For each i, let

$$\check{U}_{i,0} = U_i \times \{i\} \times \widehat{T}_{i,0} , \quad \widetilde{U}_{i,0} = \check{U}_{i,0} \cap \widetilde{X}_0 , \quad \widehat{U}_{i,0} = q(\widetilde{U}_{i,0}) .$$

Lemma 5.1.1. *$\widehat{U}_{i,0}$ is open in $\widehat{X}_0$.*

Proof. We have to check that $q^{-1}(\widehat{U}_{i,0}) \cap \widetilde{U}_{j,0}$ is open in $\widetilde{U}_{j,0}$ for all j, which is true because

$$q^{-1}(\widehat{U}_{i,0}) \cap \widetilde{U}_{j,0} = ((U_i \cap U_j) \times \{i\} \times \widehat{T}_{i,0}) \cap \widetilde{X}_0 . \quad \square$$

Lemma 5.1.2. *$q : \widetilde{U}_{i,0} \to \widehat{U}_{i,0}$ is a homeomorphism.*

Proof. This map is surjective by the definition of $\widehat{U}_{i,0}$. On the other hand, two equivalent triples in $\widetilde{U}_{i,0}$ are of the form (x, i, γ) and (x, i, δ) with $\gamma = \widehat{h_{ii}}(\delta) = \delta$ because $h_{ii} = \mathrm{id}_{T_i}$. So $q : \widetilde{U}_{i,0} \to \widehat{U}_{i,0}$ is also injective. Since $q : \widetilde{U}_{i,0} \to \widehat{U}_{i,0}$ is continuous, it only remains to prove that this map is open. A basis of the topology of $\widehat{U}_{i,0}$ consists of the sets of the form $(V \times \{i\} \times W) \cap \widetilde{X}_0$, where V and W are open in U_i and $\widehat{T}_{i,0}$ respectively. For this type of sets, we will show that

$$\begin{aligned} &\widetilde{U}_{j,0} \cap q^{-1}q\left((V \times \{i\} \times W) \cap \widetilde{X}_0\right) \\ &\qquad = \widetilde{U}_{j,0} \cap \left(V \times \{j\} \times \widehat{h_{ij}}(W \cap \operatorname{dom} \widehat{h_{ij}})\right) \end{aligned} \tag{5.2}$$

for all j, which is open in $\widetilde{U}_{j,0}$. So $q^{-1}q((V \times \{i\} \times W) \cap \widetilde{X}_0)$ is open in $\widetilde{X}_0$ and therefore $q((V \times \{i\} \times W) \cap \widetilde{X}_0)$ is open in $\widehat{X}_0$. The equality (5.2) holds because, for $(x, i, \gamma) \sim (y, j, \delta)$, with $x \in V$, $\gamma \in W$, $y \in U_j$ and $\delta \in \widehat{T}_{j,0}$, we have $y = x \in V$ and $\widehat{h_{ji}}(\delta) = \gamma \in W$. $\square$

Proposition 5.1.3. *$\widehat{X}_0$ is compact and Polish.*

Proof. Let (U'_i, p'_i, h'_{ij}) be a shrinking of (U_i, p_i, h_{ij}); i.e., it is a defining cocycle of $\mathcal{F}$ such that $\overline{U'_i} \subset U_i$ and $p'_i : U'_i \to T'_i$ is the restricion of p_i for all i. Therefore each h'_{ij} is also a restriction of h_{ij} and T'_i is a relatively locally compact open subset of T_i. Then $\hat{\pi}_0^{-1}(\overline{T'_i})$ is a compact subset of $\widehat{T}_{i,0}$ by Corollary 4.4.3. Moreover $\widehat{X}_0$ is the union of the sets $q(\overline{U'_i} \times \{i\} \times \hat{\pi}_0^{-1}(\overline{T'_i}))$. So $\widehat{X}_0$ is compact because it is a finite union of compact sets.

On the other hand, since $\widetilde{X}_0$ is closed in $\check{X}_0$, and $\check{U}_{i,0}$ is Polish and locally compact by Corollary 4.4.1, it follows that $\widehat{U}_{i,0}$ is Polish and locally compact by Lemmas 5.1.1 and 5.1.2. Then, by the compactness of $\widehat{X}_0$ and [29, Theorem 5.3], it only remains to prove that $\widehat{X}_0$ is Hausdorff.

Let $[x,i,\gamma] \neq [y,j,\delta]$ in $\widehat{X}_0$. So $x \in U_i$ and $y \in U_j$. If $x = y$, then $[y,j,\delta] = [x,i,\widehat{h_{ji}}(\delta)] \in \widehat{U}_{i,0}$. Thus, in this case, $[x,i,\gamma]$ and $[y,j,\delta]$ can be separated by open subsets of $\widehat{U}_{i,0}$ because $\widehat{U}_{i,0}$ is Hausdorff.

Now suppose that $x \neq y$. Then take disjoint open neighborhoods, V of x in U_i and W of y in U_j. Let

$$\begin{aligned} \check{V} &= V \times \{i\} \times \widehat{T}_{i,0} \subset \check{U}_{i,0} \ , \\ \check{W} &= V \times \{j\} \times \widehat{T}_{j,0} \subset \check{U}_{j,0} \ , \\ \widetilde{V} &= \check{V} \cap \widetilde{X}_0 \subset \widetilde{U}_{i,0} \ , \\ \widetilde{W} &= \check{W} \cap \widetilde{X}_0 \subset \widetilde{U}_{j,0} \ , \\ \widehat{V} &= q(\widetilde{V}) \subset \widehat{U}_{i,0} \ , \\ \widehat{W} &= q(\widetilde{W}) \subset \widehat{U}_{j,0} \ . \end{aligned}$$

$\widehat{V}$ and $\widehat{W}$ are open neighborhoods of $[x,i,\gamma]$ and $[y,j,\delta]$ in $\widehat{X}_0$. Suppose that $\widehat{V} \cap \widehat{W} \neq \emptyset$. Then there is a point $(x',i,\gamma') \in \widetilde{V}$ equivalent to some point $(y',j,\delta') \in \widetilde{W}$. This implies that $x' = y' \in V \cap W$, which is a contradiction because $V \cap W = \emptyset$. Therefore $\widehat{V} \cap \widehat{W} = \emptyset$. $\square$

According to the above equivalence relation of triples, a map $\hat{\pi}_0 : \widehat{X}_0 \to X$ is defined by $\hat{\pi}_0([x,i,\gamma]) = x$.

Proposition 5.1.4. *The map $\hat{\pi}_0 : \widehat{X}_0 \to X$ is continuous and surjective, and its fibers are homeomorphic to each other.*

Proof. We have $\hat{\pi}_0(\widehat{U}_{i,0}) = U_i$, and the composition

$$\widetilde{U}_{i,0} \xrightarrow{\ q\ } \widehat{U}_{i,0} \xrightarrow{\ \hat{\pi}_0\ } U_i \ ,$$

is the restriction of the first factor projection $\check{U}_{i,0} \to U_i$, $(x,i,\gamma) \mapsto x$. So $\hat{\pi}_0 : \widehat{X}_0 \to X$ is continuous by Lemmas 5.1.1 and 5.1.2.

The surjectivity of $\hat{\pi}_0 : \widehat{X}_0 \to X$ follows easily from the surjectivity of each map $\hat{\pi}_0 : \widehat{T}_{i,0} \to T_i$.

Let $x \in U_i$. By Lemma 5.1.2, the fiber $\hat{\pi}_0^{-1}(x)$ can be identifed to

$$\{x\} \times \{i\} \times \hat{\pi}_0^{-1}(p_i(x)) \equiv \hat{\pi}_0^{-1}(p_i(x)) \subset \widehat{T}_{i,0}.$$

So the last asertion of the statement follows from Proposition 4.4.4. $\square$

Let $\tilde{p}_{i,0} : \widetilde{U}_{i,0} \to \widehat{T}_{i,0}$ denote the restriction of the third factor projection $\check{p}_{i,0} : \check{U}_{i,0} = U_i \times \{i\} \times \widehat{T}_{i,0} \to \widehat{T}_{i,0}$. By Lemma 5.1.2, $\tilde{p}_{i,0}$ induces a continuous map $\hat{p}_{i,0} : \widehat{U}_{i,0} \to \widehat{T}_{i,0}$.

Proposition 5.1.5. $\{\widehat{U}_{0.i},\hat{p}_{i,0},\widehat{h_{ij}}\}$ *is a defining cocycle of a foliated structure* $\widehat{\mathcal{F}}_0$ *on* $\widehat{X}_0$ *induced by a regular foliated atlas.*

Proof. Let $\{U_i,\phi_i\}$ be a regular foliated atlas of $\mathcal{F}$ inducing the defining cocycle $\{U_i,p_i,h_{ij}\}$, where $\phi_i:U_i\to B_i\times T_i$ is a homeomorphism and B_i is a ball in R^n ($n=\dim\mathcal{F}$). Then we get a homeomorphism

$$\check{\phi}_{i,0}=\phi_i\times\mathrm{id}:\check{U}_{i,0}=U_i\times\{i\}\times\widehat{T}_{i,0}\to B_i\times T_i\times\{i\}\times\widehat{T}_{i,0}\ .$$

Observe that $\check{\phi}_{i,0}(\widetilde{U}_{i,0})$ consists of the elements (y,z,i,γ) with $\hat{\pi}_0(\gamma)=z$. So $\check{\phi}_{i,0}$ restricts to a homeomorphism

$$\tilde{\phi}_{i,0}:\widetilde{U}_{i,0}\to\check{\phi}_{i,0}(\check{U}_{i,0})\equiv B_i\times\{i\}\times\widehat{T}_{i,0}\equiv B_i\times\widehat{T}_{i,0}\ .$$

By Lemma 5.1.2, $\tilde{\phi}_{i,0}$ induces a homeomorphism $\hat{\phi}_{i,0}:\widehat{U}_{i,0}\to B_i\times\widehat{T}_{i,0}$. Moreover, $\check{p}_{i,0}$ corresponds to the last factor projection via $\check{\phi}_{i,0}$, obtaining that $\tilde{p}_{i,0}$ corresponds to the second factor projection via $\tilde{\phi}_{i,0}$, and therefore $\hat{p}_{i,0}$ also corresponds to the second factor projection via $\hat{\phi}_{i,0}$. Observe that (5.1) means that $\hat{p}_{i,0}=\widehat{h_{ji}}\hat{p}_{j,0}$ on $\widehat{U}_{i,0}\cap\widehat{U}_{j,0}$. The regularity of the foliated atlas $\{\widehat{U}_{0.i},\hat{\phi}_{i,0}\}$ follows easily from the regularity of $\{U_i,\phi_i\}$. □

According to Proposition 5.1.5, the holonomy pseudogroup of $\widehat{\mathcal{F}}_0$ is represented by the pseudogroup on $\bigsqcup_i\widehat{T}_{i,0}$ generated by the maps $\widehat{h_{ij}}$, which is the pseudogroup $\widehat{\mathcal{H}}_0$ on $\widehat{T}_0$.

Corollary 5.1.6. $\widehat{\mathcal{F}}_0$ *is equicontinuous and minimal.*

Proof. This follows directly from Proposition 5.1.5, and Corollaries 4.5.12 and 4.5.13. □

Corollary 5.1.7. *There is some locally compact, metrizable and separable local group* G *and some dense finitely generated sub-local group* $\Gamma\subset G$ *such that the holonomy pseudogroup of* $\widehat{\mathcal{F}}_0$ *can be represented by the pseudogroup generated by the local action of* Γ *on* G *by local left translations.*

Proof. This follows directly from Propositions 5.1.5 and 4.5.16. □

Proposition 5.1.8. $\hat{\pi}_0:(\widehat{X}_0,\widehat{\mathcal{F}}_0)\to(X,\mathcal{F})$ *is a foliated map.*

Proof. According to Proposition 5.1.5, this follows by checking the commutativity of each diagram

$$\begin{array}{ccc}\widehat{U}_{i,0} & \xrightarrow{\hat{p}_{i,0}} & \widehat{T}_{i,0}\\ {\scriptstyle\hat{\pi}_0}\downarrow & & \downarrow{\scriptstyle\hat{\pi}_0}\\ U_i & \xrightarrow{p_i} & T_i\end{array}$$

By Lemma 5.1.2, and the definition of $\hat{p}_{i,0}$ and $\hat{\pi}_{i,0}$, this commutativity follows from the commutativity of

$$\begin{array}{ccc} \widetilde{U}_{i,0} & \xrightarrow{g} & \widehat{T}_{i,0} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\pi}_0} \\ U_i & \xrightarrow{p_i} & T_i \end{array}$$

where f and g denote the restrictions of the first and third factor projections of $\check{U}_{i,0} = U_i \times \{i\} \times \widehat{T}_{i,0}$. But the commutativity of this diagram is a consequence of the definition of $\widetilde{X}_0$ and $\widetilde{U}_{i,0}$. □

Proposition 5.1.9. *The restrictions of $\hat{\pi}_0 : \widehat{X}_0 \to X$ to the leaves are the holonomy covers of the leaves of $\mathcal{F}$.*

Proof. With the notation of the proof of Proposition 5.1.5, the diagram

$$\begin{array}{ccc} \widehat{U}_{i,0} & \xrightarrow{\hat{\phi}_{i,0}} & B_i \times \widehat{T}_{i,0} \\ {\scriptstyle \hat{\pi}_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{B_i} \times \hat{\pi}_0} \\ U_i & \xrightarrow{\phi_i} & B_i \times T_i \end{array}$$

is commutative, and $\widehat{U}_{i,0} = \hat{\pi}_0^{-1}(U_i)$. Hence, for the plaques in U_i and $\widehat{U}_{i,0}$, $P_z = \phi_0^{-1}(B_i \times \{\hat{z}\})$ and $\widehat{P}_{\hat{z}} = \hat{\phi}_0^{-1}(B_i \times \{z\})$ with $z \in T_i$ and $\hat{z} \in \hat{\pi}_0^{-1}(z) \subset \widehat{T}_{i,0}$, the restriction $\hat{\pi}_0 : \widehat{P}_{\hat{z}} \to P_z$ is a homeomorphism. It follows easily that $\hat{\pi}_0 : \widehat{X}_0 \to X$ restrics to covering maps of the leaves of $\widehat{\mathcal{F}}_0$ to the leaves of $\mathcal{F}$. In fact, these are the holonomy covers, which can be seen as follows.

According to the proof of Proposition 5.1.4 and the definition of equivalence relation "$\sim$" on $\widetilde{X}_0$, for each x in $U_i \cap U_j$, we have homeomorphisms

$$\hat{\pi}_0^{-1}(p_i(x)) \xleftarrow{\hat{p}_{i,0}} \hat{\pi}_0^{-1}(x) \xrightarrow{\hat{p}_{j,0}} \hat{\pi}_0^{-1}(p_j(x))$$

satisfying $\hat{p}_{j,0}\hat{p}_{i,0}^{-1} = \widehat{h_{ij}}$. This easily implies the following. Given $x \in U_i$ and $\hat{x} \in \hat{\pi}_0^{-1}(X)$, denoting by L and $\widehat{L}$ the leaves through x and $\hat{x}$, respectively, and given a loop c in L based at x inducing a local holonomy transformation $h \in S$ around $p_i(x)$ in T_i, the lift $\hat{c}$ of c to $\widehat{L}$ with $\hat{c}(0) = \hat{x}$ satisfies $\hat{p}_{i,0}\hat{c}(1) = \hat{h}\hat{p}_{i,0}(\hat{x})$. Writing $\hat{p}_{i,0}(\hat{x}) = \boldsymbol{\gamma}(f, p_i(x))$, we obtain

$$\hat{p}_{i,0}\hat{c}(1) = \hat{h}(\boldsymbol{\gamma}(f, p_i(x))) = \boldsymbol{\gamma}(fh, p_i(x)) .$$

Therefore $\hat{c}$ is a loop if and only if $\boldsymbol{\gamma}(fh, p_i(x)) = \boldsymbol{\gamma}(f, p_i(x))$, which means $\boldsymbol{\gamma}(h, p_i(x)) = \boldsymbol{\gamma}(\mathrm{id}_T, p_i(x))$. So $\widehat{L}$ is the holonomy cover of L. □

Theorem A is the combination of the results of this section.

5.2 Independence of the choices involved

Let x_1 be another point of X, and let $\widehat{X}_1$, $\widehat{\mathfrak{F}}_1$ and $\hat{\pi}_1 : \widehat{X}_1 \to X$ be constructed like $\widehat{X}_0$, $\widehat{\mathfrak{F}}_0$ and $\hat{\pi}_0 : \widehat{X}_0 \to X$ by using x_1 instead of x_0.

Proposition 5.2.1. *There is a foliated homeomorphism $\hat{\theta} : (\widehat{X}_0, \widehat{\mathfrak{F}}_0) \to (\widehat{X}_1, \widehat{\mathfrak{F}}_1)$ such that $\hat{\pi}_1 F = \hat{\pi}_0$.*

Proof. Take an index i_1 such that $x_i \in U_{i_1}$. Let $\widehat{S}_1$, $\widehat{T}_1$, $\widehat{\mathcal{H}}_1$ and $\hat{\pi}_1 : \widehat{T}_1 \to T$ be constructed like $\widehat{S}_0$, $\widehat{T}_0$, $\widehat{\mathcal{H}}_0$ and $\hat{\pi}_0 : \widehat{T}_0 \to T$ by using $p_{i_1}(x_1)$ instead of $p_{i_0}(x_0)$, and let $\widehat{T}_{i,1} = \hat{\pi}_1^{-1}(T_i)$. Then the construction of $\widehat{X}_1$, $\widehat{\mathfrak{F}}_1$ and $\hat{\pi}_1 : \widehat{X}_1 \to X$ involves the objects $\check{X}_1$, $\widetilde{X}_1$, $\check{U}_{i,1}$, $\widetilde{U}_{i,1}$, $\widehat{U}_{i,1}$, $\check{p}_{i,1}$, $\tilde{p}_{i,1}$, $\hat{p}_{i,1}$, $\check{\phi}_{i,1}$, $\tilde{\phi}_{i,1}$ and $\hat{\phi}_{i,1}$, defined like $\check{X}_0$, $\widetilde{X}_0$, $\check{U}_{i,0}$, $\widetilde{U}_{i,0}$, $\widehat{U}_{i,0}$, $\check{p}_{i,0}$, $\tilde{p}_{i,0}$, $\hat{p}_{i,0}$, $\check{\phi}_{i,0}$, $\tilde{\phi}_{i,0}$ and $\hat{\phi}_{i,0}$, by using $\widehat{T}_{i,1}$ and $\hat{\pi}_1 : \widehat{T}_{i,1} \to T_i$ instead of $\widehat{T}_{i,0}$ and $\hat{\pi}_0 : \widehat{T}_{i,0} \to T_i$. Let $\theta : \widehat{T}_0 \to \widehat{T}_1$ be the homeomorphism given by Proposition 4.6.1, which obviously restricts to homeomorphisms $\theta_i : \widehat{T}_{i,0} \to \widehat{T}_{i,1}$. Since $\hat{\pi}_0 = \hat{\pi}_1\theta$, it follows that each homeomorphism

$$\check{\theta}_i = \mathrm{id}_{U_i \times \{i\}} \times \theta_i : \check{U}_{i,0} = U_i \times \{i\} \times \widehat{T}_{i,0} \to \check{U}_{i,1} = U_i \times \{i\} \times \widehat{T}_{i,1}$$

restricts to a homeomorphism $\tilde{\theta}_i = \widetilde{U}_{i,0} \to \widetilde{U}_{i,1}$. The combination of the homeomorphisms $\tilde{\theta}_i$ is a homeomorphism $\tilde{\theta} : \widetilde{X}_0 \to \widetilde{X}_1$.

We can assume that the pseudo$*$group S used in the definition of $\widehat{T}_0$, $\widehat{\mathcal{H}}_0$, $\widehat{T}_1$ and $\widehat{\mathcal{H}}_1$is the localization of the pseudo$*$group generated by the maps h_{ij}. For each $h \in S$, use the notation $\hat{h}_0 \in \widehat{S}_0$ and $\hat{h}_1 \in \widehat{S}_1$ for the map $\hat{h}$ defined with $p_{i_0}(x_0)$ and $p_{i_1}(x_1)$, respectively. From the proof of Proposition 4.6.1, we get $\hat{h}_1\theta = \theta\hat{h}_0$ for all $h \in S$; in particular, this holds with $h = h_{ij}$. So $\tilde{\theta} : \widetilde{X}_0 \to \widetilde{X}_1$ is compatible with the equivalence relations used to define $\widehat{X}_0$ and $\widehat{X}_1$, and therefore it induces a homeomorphism $\hat{\theta} : \widehat{X}_0 \to \widehat{X}_1$. Note that $\hat{\theta}$ restricts to homeomorphisms $\hat{\theta}_i : \widehat{U}_{i,0} \to \widehat{U}_{i,1}$. Obviously, $\check{p}_{i,1}\check{\theta}_i = \theta_i\check{p}_{i,1}$, yielding $\tilde{p}_{i,1}\tilde{\theta}_i = \theta_i\tilde{p}_{i,1}$, and therefore $\hat{p}_{i,1}\hat{\theta}_i = \theta_i\hat{p}_{i,1}$. It follows that $\hat{\theta}$ is a foliated map. □

Let $\{U'_a, p'_a, h'_{ab}\}$ be another defining cocycle of $\mathcal{F}$ induced by a regular foliated atlas. Then let $\widehat{X}'_0$, $\widehat{\mathfrak{F}}'_0$ and $\hat{\pi}'_0 : \widehat{X}_1 \to X$ be constructed like $\widehat{X}_0$, $\widehat{\mathfrak{F}}_0$ and $\hat{\pi}_0 : \widehat{X}_0 \to X$ by using $\{U'_a, p'_a, h'_{ab}\}$ instead of $\{U_i, p_i, h_{ij}\}$.

Proposition 5.2.2. *There is a foliated homeomorphism $F : (\widehat{X}_0, \widehat{\mathfrak{F}}_0) \to (\widehat{X}'_0, \widehat{\mathfrak{F}}'_0)$ such that $\hat{\pi}'_0 F = \hat{\pi}_0$.*

Proof. By using a common refinement of the open coverings $\{U_i\}$ and $\{U'_a\}$, we can assume that $\{U'_a\}$ refines $\{U_i\}$. In this case, the union of the defining cocycles $\{U_i, p_i, h_{ij}\}$ and $\{U'_a, p'_a, h'_{ab}\}$ is contained in another defining cocycle induced by a regular foliated atlas. Thus the proof boils down to show that a sub-defining cocycle $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ of $\{U_i, p_i, h_{ij}\}$ induces a foliated space homeomorphic to $(\widehat{X}_0, \widehat{\mathcal{F}}_0)$. But the pseudogroup $\mathcal{H}'$ induced by $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ is the restriction of $\mathcal{H}$ to an open subset $T' \subset T$, and the pseudo$*$group induced by $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ is $S' = S \cap \mathcal{H}'$. Then, by using the canonical identity given by Proposition 4.6.3, it easily follows that the foliated space $(\widehat{X}'_0, \widehat{\mathcal{F}}'_0)$ defined with $\{U_{i_k}, p_{i_k}, h_{i_k i_l}\}$ can be canonically identified with an open foliated subspace of $(\widehat{X}_0, \widehat{\mathcal{F}}_0)$, which indeed is the whole of $(\widehat{X}_0, \widehat{\mathcal{F}}_0)$ because $\{U_{i_k}\}$ covers X. $\square$

Chapter 6

Growth of equicontinuous foliated spaces

Contents

6.1 Growth of equicontinuous pseudogroups and foliated spaces

Let G be a locally compact Polish local group with a left-invariant metric, let $\Gamma \subset G$ be a dense finitely generated sub-local group, and let $\mathcal{H}$ denote the pseudogroup generated by the local action of Γ on G by local left translations. Suppose that $\mathcal{H}$ is compactly generated. Let $\mathcal{G} = \mathcal{H}|_U$ for some relatively compact open identity neighborhood U in G, which meets all $\mathcal{H}$-orbits because Γ is dense. Let E be a recurrent symmetric system of compact generation of $\mathcal{H}$ on U. Let $\mathfrak{G}$ be the groupoid of germs of maps in $\mathcal{G}$. It will be said that *G can be approximated by nilpotent local Lie groups* if, there is a sequence F_n given by Theorem 3.4.3 so that the local Lie groups $G/(F_n, U)$ are nilpotent.

Theorem 6.1.1. *With the above notation and conditions, one of the following properties hold:*

- *G can be approximated by nilpotent local Lie groups; or*
- *the germ covers of all $\mathcal{G}$-orbits have exponential growth with d_E.*

Proof. According to Theorem 3.4.3, there is some $U_0 \in \Psi G$, contained in any given element of $\Psi G \cap \Phi(G, 2)$, and there exists a sequence of compact normal subgroups $F_n \subset U_0$ such that $F_{n+1} \subset F_n$, $\bigcap_n F_n = \{e\}$, $(F_n, U_0) \in \Delta G$, and $G/(F_n, U_0)$ is a local Lie group. Let $T_n : U_0^2 \to G/(F_n, U_0)$ denote the canonical projection. Take some open identity neighborhood U_1 such that $\overline{U_1} \subset U_0$. Then $F_n\overline{U_1} \subset U_0$ for n large enough by the properties of the sequence F_n. Let $U_2 = F_nU_1$ for such an n. Thus U_2 is saturated by the fibers of T_n, and $\overline{U_2} \subset U_0$. Then $U' = T_n(U_2)$ is a relatively compact open identity neighborhood in the local Lie group $G' = G/(F_n, U_0)$. Let $\Gamma' = T_n(\Gamma \cap U_0^2)$, which is a dense sub-local group of G', and let $\mathcal{H}'$ denote the pseudogroup on G' generated by the local action of Γ' by local left translations.

For every $\gamma \in \Gamma \cap U_0$ with $\gamma U_2 \cap U_2 \neq \emptyset$, let h_γ denote the restriction $U_2 \cap \gamma^{-1} U_2 \to \gamma U_2 \cap U_2$ of the local left translation by γ. There is a finite symmetric set $S = \{s_1, \dots, s_k\} \subset \Gamma$ such that $E_2 = \{h_{s_1}, \dots, h_{s_k}\}$ is a recurrent system of compact generation of $\mathcal{H}$ on U_2. By reducing Γ if necessary, we can suppose that S generates Γ. For every $\delta \in \Gamma'$ with $\delta U' \cap U' \neq \emptyset$, let h'_δ denote the restriction $U' \cap \delta^{-1} U' \to \delta U' \cap U'$ of the local left translation by δ. We can assume that $s_1, \dots, s_k$ are in U_2, and therefore we can consider their images $s'_1, \dots, s'_k$ by T_n. Moreover each h_{s_i} induces via T the map h'_{s_i}, and $E' = \{h'_{s_1}, \dots, h'_{s_k}\}$ is a system of compact generation of $\mathcal{H}'$ on U'. By increasing E_2 if necessary, we can assume that E' is also recurrent. Fix any open set V' in G' with $\overline{V'} \subset U'$. Then $V = T_n^{-1}(V')$ satisfies $\overline{V} \subset U_2$.

Claim 6. For each finite subset $F \subset \Gamma \cap U_2$, we have

$$U_2 \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma V \ .$$

Since U_2 and V are saturated by the fibers of T_n, Claim 6 follows by showing that

$$U' \subset \bigcup_{\gamma \in \Gamma' \setminus F'} \gamma V' \ , \tag{6.1}$$

where $F' = T_n(F)$. Suppose that (6.1) is false. Then there is some finite symmetric subset $F \subset \Gamma \cap U_2$ and some $x \in U'$ such that $((\Gamma' \setminus F')x) \cap V' = \emptyset$. By the recurrence of E', there is some $N \in \mathbb{N}$ satisfying (3.1) with U' and E'. Since $\Gamma'_{U',x}$ is infinite because Γ' is dense in G', it follows that there is some $\gamma \in \Gamma'_{U',x} \setminus F'$ such that

$$|\gamma|_{S',U',x} > N + \max\{\, |\varepsilon|_{S',U',x} \mid \varepsilon \in F' \cap \Gamma'_{U',x} \,\} \ . \tag{6.2}$$

By (3.1), there is some $h \in E'^N$ such that

$$\gamma x \in h^{-1}(V' \cap \operatorname{im} h') \ .$$

We have $h = h'_\delta$ for some $\delta \in \Gamma'$. Note that $\delta \in \Gamma'_{U',\gamma' x}$ and $|\delta|_{S',U',\gamma x} \leq N$. Hence

$$\begin{aligned} |\gamma|_{S',U',x} &\leq |\delta\gamma|_{S',U',x} + |\delta^{-1}|_{S',U',\delta\gamma x} \\ &= |\delta\gamma|_{S',U',x} + |\delta|_{S',U',\gamma' x} \leq |\delta\gamma|_{S',U',x} + N \end{aligned}$$

by (3.2) and (3.3), obtaining that $\delta\gamma \notin F'$ by (6.2). However, $\delta\gamma x \in V'$, obtaining a contradiction, which completes the proof of Claim 6.

Claim 7. For each finite subset $F \subset \Gamma \cap U_2$, we have

$$\overline{U_2} \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma V \, .$$

Take a relatively compact open subset $O_1 \subset G$ such that $\overline{U_1} \subset O_1$ and $F_n \overline{O_1} \subset U_0$. Let $O_2 = F_n O_1$ and $\mathcal{K} = \mathcal{H}|_{O_2}$. Then Claim 7 follows by applying Claim 6 to O_2.

According to Claim 7, by increasing S if necessary, we can suppose that

$$\overline{U_2} \subset \bigcup_{i<j} (s_i \cdot V \cap s_j \cdot V) = \bigcup_{i<j} (s_i^{-1} \cdot V \cap s_j^{-1} \cdot V) \, . \tag{6.3}$$

Suppose that G cannot be approximated by nilpotent local Lie groups. Then we can assume that the local Lie group G' is not nilpotent. Moreover we can suppose that G' is a sub-local Lie group of a simply connected Lie group L. Let Δ be the dense subgroup of L whose intersection with G' is Γ'. Then, by Proposition 3.11.1, there are elements $t'_1, \dots, t'_k$ in Δ, as close as desired to $s'_1, \dots, s'_k$, which are free generators of a free semi-group. If the elements t'_i are close enough to s'_i, then they are in U'. So there are elements $t_i \in U_2$ such that $T_n(t_i) = t'_i$. By the compactness of $\overline{U_2}$, and because U_2 and V are saturated by the fibers of T_n, if $t'_1, \dots, t'_k$ are close enough to $s'_1, \dots, s'_k$, then (6.3) gives

$$\overline{U_2} \subset \bigcup_{i<j} (t_i^{-1} V \cap t_j^{-1} V) \, . \tag{6.4}$$

Now, we adapt the argument of the proof of [11, Lemma 10.6]. Let $\widehat{\Gamma} \subset \Gamma$ be the sub-local group generated by $t_1, \dots, t_k$; thus $\widehat{S} = \{t_1^{\pm 1}, \dots, t_k^{\pm 1}\}$ is a symmetric set of generators of $\widehat{\Gamma}$, and $S \cup \widehat{S}$ is a symmetric set of generators of Γ. With $\widehat{E} = \{h_{t_1}^{\pm 1}, \dots, h_{t_k}^{\pm 1}\}$, observe that $E_2 \cup \widehat{E}$ is a recurrent system of compact generation of $\mathcal{H}$ on U_2. Given $x \in U_2$, let $\mathsf{S}(n)$ be the sphere with center e and radius $n \in \mathbb{N}$ in $\widehat{\Gamma}_{U_2,x}$ with $d_{\widehat{S},U,x}$. By (6.4), for each $\gamma \in \mathsf{S}(n)$, there are indices $i < j$ such that $\gamma x \in t_i^{-1} V \cap t_j^{-1} V$. So the points $t_i \gamma x$ and $t_j \gamma x$ are in V, obtaining that $t_i \gamma, t_j \gamma \in \mathsf{S}(n+1)$. Moreover all elements obtained in this way from elements of $\mathsf{S}(n)$ are pairwise distinct because $t'_1, \dots, t'_k$ freely generate a free semigroup. Hence $\operatorname{card}(\mathsf{S}(n+1)) \geq 2 \operatorname{card}(\mathsf{S}(n))$, giving $\operatorname{card}(\mathsf{S}(n)) \geq 2^n$. So $\widehat{\Gamma}_{U_2,x}$ has exponential growth with $d_{\widehat{S},U_2,x}$. Since $\widehat{\Gamma}_{U_2,x} \subset \Gamma_{U_2,x}$ and $d_{S \cup \widehat{S},U_2,x} \leq d_{\widehat{S},U_2,x}$ on $\widehat{\Gamma}_{U_2,x}$, it follows that $\Gamma_{U_2,x}$ also has exponential growth with $d_{S \cup \widehat{S},U_2,x}$. So $\mathfrak{G}_x$ has exponential growth with $d_{E_2 \cup \widehat{E}}$, obtaining that $\mathfrak{G}_x$ has exponential growth with d_E by Corollary 3.9.4. $\square$

Theorem B follows from Theorem 6.1.1 and the observations of Section 3.10.

CHAPTER 7

Examples

Contents

Many examples of foliated spaces are given by Candel and Conlon [12, Chapter 11], including some of the examples considered here.

7.1 Locally free actions

Any locally free action of a connected Lie group on a locally compact Polish space, $\phi : H \times X \to X$, defines a foliated structure $\mathcal{F}$ on X whose leaves are the orbits (see [12, Theorem 11.3.14], [37]). Let $\mathcal{H}$ denote the holonomy pseudogroup of $\mathcal{F}$. Let $\operatorname{Homeo}(X)$ denote the topological group of homeomorphisms of X with the compact-open topology. Suppose that X is compact and ϕ equicontinuous. Then $\mathcal{F}$ is also minimal and equicontinuous. Moreover the closure $\widehat{H} = \overline{\{\, \phi_h \mid h \in H \,\}}$ in $\operatorname{Homeo}(X)$ is compact, and its canonical action on X is transitive. Hence X can be identified with a homogeneous space of $\widehat{H}$.

Question 7.1.1. Relate $\overline{\mathcal{H}}$ to $\widehat{H}$; in particular, use $\widehat{H}$ to describe the strong quasi-analyticy of $\overline{\mathcal{H}}$ and the foliated space $\widehat{X}_0$ given by Theorem A.

Any oriented foliated space X of dimension one is defined by a non-singular flow $\phi : \mathbb{R} \times X \to X$. To see this, use a partition of unity subordinated to a foliated atlas to produce a measure on the leaves, positive on non-empty open sets. Then the action ϕ is easy to define by using this measure and the orientation.

7.2 Matchbox manifolds and solenoids

A *matchbox manifold* is a foliated continuum[1] $X \equiv (X, \mathcal{F})$ transversely modeled on a totally disconnected space. In this case, the foliated structure $\mathcal{F}$ is determined by the topology of X: the leaves are the path connected components of X. Since X is connected, either X is a compact manifold, or it is transversely modeled on a Cantor space. It is said that X is C^k ($k \in \mathbb{N}$) if the changes of foliated coordinates are leafwise C^k, with transversely continuous leafwise derivatives of order $\leq k$; if X is C^k for all k, then it is said that X is C^∞ or *smooth*. It will be assumed that all matchbox manifolds are C^1.

The classical solenoids over the circle are examples of matchbox manifolds. With more generality, an n-dimensional *solenoid* is an inverse limit

$$\mathcal{S} = \varprojlim \{p_{l+1} : L_{l+1} \to L_l\} ,$$

where each L_l ($l \in \mathbb{N}$) is a closed connected n-dimensional manifold, and the maps $p_{l+1} : L_{l+1} \to L_l$ are smooth proper covering maps. Any n-dimensional solenoid is a matchbox manifold transversely modeled on a Cantor space. If any composite of a finite number of bounding maps p_l is a normal covering, then $\mathcal{S}$ is called a *McCord solenoid*.

It is easy to check that any solenoid is equicontinuous. The reciprocal was shown by Clark-Hurder [16, Theorem 7.9]: any equicontinuous matchbox manifold is a solenoid. More generally, Alcalde-Lozano-Macho [2] have shown that any minimal transversely Cantor n-dimensional matchbox manifold is an inverse limit of compact branched n-manifolds.

On the other hand, recall that a topological space (respectively, foliated space) is called *homogeneous* when the canonical action of its group of homeomorphisms (respectively, foliated homeomorphisms) is transitive. For matchbox manifolds, the homogeneities as topological and foliated space are equivalent. It is easy to check that McCord solenoids are homogeneous. The reciprocal was also proved by Clark-Hurder [16, Theorem 1.1]: any homogeneous matchbox manifold is homeomorphic to a McCord solenoid; in particular, it is minimal.

For a McCord solenoid $\mathcal{S}$ as above, let

$$\cdots \longrightarrow H_{l+1} \xrightarrow{p_{l+1\,*}} H_l \longrightarrow \cdots$$

denote the corresponding sequence of injective homomorphisms between fundamental groups. We get a sequence of canonical projections between finite groups,

$$\cdots \longrightarrow G_{l+1} = H_0/p_{l+1\,*}(H_{l+1}) \longrightarrow G_l = H_0/p_{l\,*}(H_l) \longrightarrow \cdots ,$$

[1]Recall that a *continuum* is a non-empty compact connected metrizable space.

whose inverse limit,

$$G = \varprojlim \{G_{l+1} \to G_l\} ,$$

is a topological group homeomorphic to a Cantor space. Then the canonical map $p : \mathcal{S} \to L_0$ is a fiber bundle whose typical fiber is G, and whose restrictions to the leaves are covering maps of L_0. Moreover $\mathcal{S}$ is transversely modeled by left translations on G (it is a *G-foliated space*), and therefore the foliated space $\widehat{\mathcal{S}}$, given by Theorem A, can be identified with $\mathcal{S}$.

When the solenoid $\mathcal{S}$ is not McCord, it is still equicontinuous, and the application of Theorem A may give non-trivial information. In this case, G is defined as a projective limit of discrete finite sets, without any group structure, but there is a canonical action of H_0 on G. Let $\mathcal{H}$ denote the holonomy pseudogroup of $\mathcal{S}$.

Question 7.2.1. If $\mathcal{S}$ is not McCord, use the H_0-space G to characterize the strong quasi-analyticity of $\overline{\mathcal{H}}$, and to describe the foliated space $\widehat{\mathcal{S}}_0$ given by Theorem A.

7.3 Almost periodic functions

Consider the space of continuous bounded functions, $C_b(\mathbb{R})$, equipped with the topology of uniform convergence. For a function $f \in C_b(\mathbb{R})$ and $t \in \mathbb{R}$, let f_t denote the translation of f by t: $f_t(r) = f(r + t)$. It is said that f is *almost periodic* (in the sense Besucovich [9] and Gottschalk [20]) if the family of traslations $\{ f_t \mid t \in \mathbb{R} \}$ is equicontinuous. In this case, the closure $\mathfrak{M}_f = \overline{\{ f_t \mid t \in \mathbb{R} \}}$ in $C_b(\mathbb{R})$ is compact. A continuous nonsingular flow $\Phi : \mathbb{R} \times \mathfrak{M}_f \to \mathfrak{M}_f$ is defined by $\Phi_t(g) = g_t$. Therefore $\mathfrak{M}_f$ is foliated by the orbits of Φ. This flow Φ is equicontinuous, obtaining that the corresponding foliated space is also equicontinuous. Moreover this foliated space is transitive (the orbit of f is dense), and therefore it is minimal.

More generally, we can define in the same way a compact minimal equicontinuous foliated space $\mathfrak{M}_f$ for any bounded continuous function f on M with values in a Hilbert space, which is almost-periodic in the same sense.

This construction is universal in the following sense. As we saw (Section 7.1), for any compact Polish space X and any oriented equicontinuous minimal foliated structure $\mathcal{F}$ on X of dimension one, there is a non-singular equicontinuous flow $\phi : \mathbb{R} \times X \to X$ whose orbits are the leaves of $\mathcal{F}$. Take a sequence of real valued continuous functions h_n on X that separate points. Let (e_n) be a complete orthonormal system of a separable Hilbert space $\mathfrak{H}$. Then the function $h : X \to \mathfrak{H}$, given by

$$h(x) = \sum_n \frac{h_n(x)}{2^n \cdot \max |h_n|} \, e_n \ ,$$

is continuous and separates points, and therefore it is a topological embedding because X is compact. Given any point $x_0 \in X$, let $f \in C_b(\mathbb{R}, \mathfrak{H})$ be defined by $f(r) = h\phi_r(x_0)$. From the equicontinuity of ϕ and the compactness of X, it follows that f is almost periodic. Thus the compact minimal equicontinuous foliated space $\mathfrak{M}_f$ is defined. The mapping $\phi_t(x_0) \mapsto f_t$ defines a continuous map of the ϕ-orbit of x_0 onto the Φ-orbit of f, which extends to an equivariant continuous map $\psi : X \to \mathfrak{M}_f$. This map is injective because h separates points, and it is surjective because Φ is minimal. Hence ψ is a foliated homeomorphism since X is compact and $\mathfrak{M}_f$ Hausdorff.

With even more generality, we can make the same type of construction for almost periodic functions on any Lie group H, obtaining all compact equicontinuous foliated spaces defined by locally free H-actions.

Question 7.3.1. Characterize the strong quasi-analyticity of the closure of the holonomy pseudogroup of $\mathfrak{M}_f$, and describe the foliated space $\widehat{\mathfrak{M}}_{f,0}$ given by Theorem A.

The realization of solenoids as minimal sets of foliations is studied by Clark-Hurder [15].

7.4 Almost periodic and locally aperiodic Riemannian manifolds

For each $n \in \mathbb{Z}_+$, let $\mathcal{M}_*(n)$ denote the set[2] of isometry classes of pointed complete connected Riemannian manifolds of dimension n. The isometry class of each pointed Riemannian manifold (M, x) will be denoted by $[M, x]$. The C^∞ *topology* on $\mathcal{M}_*(n)$ is defined by requiring that the convergence $[M_i, x_i] \to [M, x]$ means that, for each compact domain $\Omega \subset M$, there are C^∞ embeddings $\phi_i : \Omega \to M_i$ for large enough i such that $\phi_i(x) = x_i$ and $\phi_i^* g_i \to g$ on Ω with respect to the C^∞ topology, where g and g_i are the metric tensors of M and M_i. The corresponding space will be denoted by $\mathcal{M}_*^\infty(n)$. Observe that $\mathcal{M}_*^\infty(1)$ is a singleton; thus we will assume that $n \geq 2$.

For each complete connected Riemannian manifold M of dimension n, there is a canonical continuous map $\iota_M : M \to \mathcal{M}_*^\infty(n)$ given by $\iota_M(x) = [M, x]$, which induces a canonical embedding $\bar{\iota}_M : \mathrm{Iso}(M)\backslash M \to \mathcal{M}_*^\infty(n)$, where $\mathrm{Iso}(M)$ denotes the group of isometries of M. The images of the maps ι_M form a partition of $\mathcal{M}_*^\infty(n)$, called its *canonical partition*. With respect to

[2]The logical problems of the definition of this set can be avoided as follows. Observe that the underlying sets of these Riemannian manifolds are equipotent to $\mathbb{R}$. Therefore we can take only structures of Riemannian n-manifolds on $\mathbb{R}$, and then $\mathcal{M}_*(n)$ is a well defined set.

this partition, the closure of the image of each ι_M in $\mathcal{M}_*^\infty(n)$ is a saturated subspace denoted by $\mathcal{M}_*^\infty(M)$.

A complete connected Riemannian manifold M, with metric tensor g, is said to be:

aperiodic if $\operatorname{Iso}(M) = \{\mathrm{id}_M\}$;

locally aperiodic if any $x \in M$ has a neighborhood V such that

$$\{\, h \in \operatorname{Iso}(M) \mid h(x) \in V \,\} = \{\mathrm{id}_M\}\ ;$$

almost periodic if for any $m \in \mathbb{N}$, $\varepsilon > 0$ and $x \in M$, there is a set H of diffeomorphisms of M such that $\sup |\nabla^k h^* g| < \varepsilon$ for all $h \in H$ and $k \leq m$, and $\{\, h(x) \mid h \in H \,\}$ is a net in M; and of

bounded geometry if it has a positive injectivity radius, and each covariant derivative of the curvature tensor, with arbitrary order, is uniformly bounded.

Observe that M is (locally) aperiodic if and only if ι_M is (locally) injective. Moreover M has bounded geometry if it is almost periodic. If M has bounded geometry then $\mathcal{M}_*^\infty(M)$ is compact by the Fundamental Theorem of Convergence Theory [38, Chapter 10, Theorem 3.3], which is essentially due to Cheeger [14]. Álvarez and Candel [6] have proved that, if the complete connected Riemannian n-manifold M is almost periodic and locally aperiodic, then $\mathcal{M}_*^\infty(M)$ becomes a compact minimal equicontinuous foliated space of dimension n with the canonical partition. In this case, we ask the following.

Question 7.4.1. Characterize the strong quasi-analyticity of the closure of the holonomy pseudogroup of $\mathcal{M}_*^\infty(M)$ in terms of M, and describe the foliated space $\widehat{\mathcal{M}_*^\infty(M)}_0$ given by Theorem A.

CHAPTER 8

Appendices

Contents

A Molino's theory for Riemannian foliations

A.1 Normal bundle

Let $\mathcal{F}$ be a smooth (C^∞) foliation of dimension p and codimension q on a manifold M. Let us recall and fix the following familiar terminology and notation. Let $T\mathcal{F} \subset TM$ and $N\mathcal{F} = TM/T\mathcal{F}$ denote the bundles of vectors tangent and normal to the leaves, called the *tangent* and *normal bundles* of $\mathcal{F}$, respectively. The frames and smooth sections of $N\mathcal{F}$ are called *normal frames* and *normal vector fields*, respectively. By composition with the canonical projection $TM \to N\mathcal{F}$, any $X \in TM$ (respectively, $X \in \mathfrak{X}(M)$) defines an element of $N\mathcal{F}$ (respectively, a section of $N\mathcal{F}$) denoted by $\overline{X}$.

The normal bundle $N\mathcal{F}$ has a flat $T\mathcal{F}$-partial connection $\nabla^{\mathcal{F}}$ given by $\nabla^{\mathcal{F}}_V\overline{X} = \overline{[V, X]}$ for $V \in \mathfrak{X}(\mathcal{F})$ and $X \in \mathfrak{X}(M)$; this will be shortly expressed by saying that $N\mathcal{F}$ is a flat $T\mathcal{F}$-vector bundle. For each path c from x to y in a leaf, the corresponding holonomy transformation h_c is defined between smooth local transversals through x and y, and its differential can be considered as an isomorphism $h_{c*} : N_x\mathcal{F} \to N_y\mathcal{F}$, called the *infinitesimal holonomy* of c; the infinitesimal holonomy h_{c*} is given by the $\nabla^{\mathcal{F}}$-parallel transport along c.

$\nabla^{\mathcal{F}}$ induces a flat $T\mathcal{F}$-partial connection on the $GL(q)$-principal bundle $\pi_P : P \to M$ of orthonormal frames of $N\mathcal{F}$. The integral submanifolds of this principal bundle connection are the leaves of a foliation $\mathcal{F}_P$, and the restriction of π_P to its leaves defines coverings of the leaves of $\mathcal{F}$; they are the infinitesimal holonomy coverings of the leaves of $\mathcal{F}$.

A.2 Foliated maps and infinitesimal transformations

Let $\mathcal{F}'$ be another smooth foliation on a manifold M'. A (smooth) *foliated map* $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is a map $\phi : M \to M'$ that maps leaves to leaves. A foliated map $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ induces a continuous map between the corresponding leaf spaces, $\bar{\phi} : M/\mathcal{F} \to M'/\mathcal{F}'$. Also, the differential $\phi_* : TM \to TM'$ restricts to a homomorphism $\phi_* : T\mathcal{F} \to T\mathcal{F}'$, obtaining an induced homomorphism $\phi_* : N\mathcal{F} \to N\mathcal{F}'$, which is compatible with the corresponding flat partial connections. Two foliated maps $\phi, \psi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ are called *tangentially* (or *leafwisely*) *homotopic* when there is a homotopy between them that is a foliated map $(M \times I, \mathcal{F} \times I) \to (M', \mathcal{F}')$, where $I = [0, 1]$ and $\mathcal{F} \times I$ is the foliation with leaves $L \times I$ for leaves L of $\mathcal{F}$. The foliated diffeomorphisms of $(M, \mathcal{F})$, also called *transformations* of $(M, \mathcal{F})$, form a group denoted by $\operatorname{Diffeo}(M, \mathcal{F})$. A smooth flow (ϕ^t) on M is called *foliated* if $\phi^t \in \operatorname{Diffeo}(M, \mathcal{F})$ for all $t \in \mathbb{R}$. Let $\operatorname{Diffeo}(\mathcal{F}) \subset \operatorname{Diffeo}(M, \mathcal{F})$ be the normal subgroup of foliated diffeomorphisms that preserve each leaf of $\mathcal{F}$, and let $\operatorname{Diffeo}_0(\mathcal{F}) \subset \operatorname{Diffeo}(\mathcal{F})$ be the normal subgroup of foliated diffeomorphisms that are tangentially homotopic to the identity map. The quotient group

$$\overline{\operatorname{Diffeo}}(M, \mathcal{F}) = \operatorname{Diffeo}(M, \mathcal{F}) / \operatorname{Diffeo}_0(\mathcal{F})$$

consists of the tangential homotopy classes of transformations of $(M, \mathcal{F})$, which can be called *transverse transformations* of $(M, \mathcal{F})$. Observe that the canonical action of $\operatorname{Diffeo}(M, \mathcal{F})$ on M induces an action of $\overline{\operatorname{Diffeo}}(M, \mathcal{F})$ on $M/\mathcal{F}$.

Let $\mathfrak{X}(\mathcal{F})$ be the family of tangent vector fields on M that are tangent to the leaves of $\mathcal{F}$; this is a Lie subalgebra and $C^\infty(M)$-submodule of $\mathfrak{X}(M)$. Its normalizer, $\mathfrak{X}(M, \mathcal{F})$, consists of the tangent vector fields whose flows are foliated, which are called *infinitesimal transformations* of $(M, \mathcal{F})$. The quotient Lie algebra $\overline{\mathfrak{X}}(M, \mathcal{F}) = \mathfrak{X}(M, \mathcal{F}) / \mathfrak{X}(\mathcal{F})$ can be identified with the linear space of the $\nabla^{\mathcal{F}}$-parallel normal vector fields (those that are invariant by all infinitesimal holonomy transformations). Thus the element of $\overline{\mathfrak{X}}(M, \mathcal{F})$ represented by any $X \in \mathfrak{X}(M, \mathcal{F})$ is denoted by $\overline{X}$.

Ignoring the subtleties about Lie groups of infinite dimension, we can consider $\mathfrak{X}(\mathcal{F})$, $\mathfrak{X}(M, \mathcal{F})$ and $\overline{\mathfrak{X}}(M, \mathcal{F})$ as the Lie algebras of the "Lie groups" $\operatorname{Diffeo}(\mathcal{F})$, $\operatorname{Diffeo}(M, \mathcal{F})$ and $\overline{\mathfrak{X}}(M, \mathcal{F})$.

A.3 Riemannian, TP and Lie foliations

It is said that $\mathcal{F}$ is *Riemannian* when there is an invariant Riemannian metric on its holonomy pseudogroup, or, equivalently, a $\nabla^{\mathcal{F}}$-flat Riemannian structure on $N\mathcal{F}$ (it is invariant by infinitesimal holonomies). This also means that there is a Riemannian metric on M, called *bundle-like*, so that $\mathcal{F}$ is locally given by Riemannian submersions (these are just the metrics that induce $\nabla^{\mathcal{F}}$-flat Riemannian structures on $N\mathcal{F}$).

It is said that $\mathcal{F}$ is *transitive* if the evaluation map $\mathrm{ev}_x : \mathfrak{X}(M, \mathcal{F}) \to T_xM$ is surjective for all $x \in M$. This means that the canonical action of $\mathrm{Diffeo}(M, \mathcal{F})$ on M is transitive. As a particular case of transitive foliation, $\mathcal{F}$ is called *transversely parallelizable* (or, shortly, *TP*) if $N\mathcal{F}$ is trivial as $\mathcal{F}$-flat bundle; i.e., there are $X_1, \dots, X_q \in \mathfrak{X}(M, \mathcal{F})$ such that $\overline{X}_1, \dots, \overline{X}_q$ is a global frame of $N\mathcal{F}$. In this case, $\overline{X}_1, \dots, \overline{X}_q$ is called a *transverse parallelism*. If moreover $\overline{X}_1, \dots, \overline{X}_q$ is a basis of a Lie subalgebra $\mathfrak{g} \subset \overline{\mathfrak{X}}(M, \mathcal{F})$, then $\mathcal{F}$ is called a ($\mathfrak{g}$-)*Lie foliation*; this $\mathfrak{g}$ is called the *structural Lie algebra* of $\mathcal{F}$. The simply connected Lie group G with Lie algebra $\mathfrak{g}$ will be also called *structural Lie group* of $\mathcal{F}$, and the term *G-Lie foliation* may be used. Observe that the restriction $\overline{\mathrm{ev}}_x : \mathfrak{g} \to N_x\mathcal{F}$ is an isomorphism for all $x \in M$. Taking the inverse of these isomorphisms and considering the canonical injection $N_x\mathcal{F}^* \to T_xM^*$, we get a $\mathfrak{g}$-valued 1-form ω on M satisfying $\ker \omega = T\mathcal{F}$ and $d\omega + \frac{1}{2}[\omega, \omega] = 0$; the existence of such ω is a characteristic property of Lie foliations.

If the leaves of $\mathcal{F}$ are dense, observe that $\overline{\mathfrak{X}}(M, \mathcal{F})$ is a finite dimensional Lie algebra. So, if $\mathcal{F}$ is minimal and TP, then it is a $\mathfrak{g}$-Lie foliation with $\mathfrak{g} = \overline{\mathfrak{X}}(M, \mathcal{F})$. Note also that any TP foliation becomes Riemannian by requiring a transverse parallelism to be orthonormal; in fact, any transitive foliation is Riemannian.

A.4 Molino's description of Riemannian foliations

The particular case of Lie foliations on compact manifolds has the following description due to Fédida.

Theorem A.1 (Fédida [17, 18]; see also [33, 34]). *Let $\mathcal{F}$ be a $\mathfrak{g}$-Lie foliation on a compact connected manifold M. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then there are:*

- *a covering map $\pi : \widetilde{M} \to M$;*
- *a fiber bundle map $D : \widetilde{M} \to G$; and*
- *a homomorphism $h : \pi_1(M) \to G$;*

such that the leaves of $\pi^*\mathcal{F}$ are the fibers of D, and D is h-equivariant; i.e., $D(\sigma \cdot \tilde{x}) = h(\sigma)\, D(\tilde{x})$ for all $\sigma \in \pi_1(M)$ and $\tilde{x} \in \widetilde{M}$.

In Theorem A.1, D is called the *developing map* of $\mathcal{F}$, h is called its *holonomy homomorphism*, and $\Gamma = \operatorname{im} h$ is called its *holonomy group*. Observe that, according to Theorem A.1, the holonomy pseudogroup of $\mathcal{F}$ is represented by the pseudogroup generated by the action of Γ on G by left translations; this is another characteristic property of Lie foliations.

Theorem A.1 can be proved as follows. We can use the $\mathfrak{g}$-valued 1-form ω to define a flat connection η on the trivial principal G-bundle $\mathrm{pr}_1 : M \times G \to M$ by

$$\eta_{(x,g)}(\xi, \zeta) = \mathrm{Ad}_{g^{-1}}\, \omega_x(\xi) + (L_{g^{-1}})_{*\,g}(\zeta) ,$$

for any $(\xi, \zeta) \in T_{(x,g)}M \times G \equiv T_xM \oplus T_gG$. Hence $\ker \eta$ is an involutive subbundle of $TM \oplus TG$, which defines a G-invariant foliation $\mathcal{G}$ on $M \times G$ ($T\mathcal{G} = \ker \eta$). Moreover the restriction of $(\mathrm{pr}_2)_{*\,(x,g)} : T_xM \oplus T_gG \to T_gG$ to $T_{(x,g)}\mathcal{G}$ is an epimorphism $T_{(x,g)}\mathcal{G} \to T_gG$ since ω_x is surjective. Hence, for any leave $\widetilde{M}$ of $\mathcal{G}$, the restriction of pr_2 to $\widetilde{M}$ is an equivariant submersion $D : \widetilde{M} \to G$. On the other hand, the restriction of pr_1 to $\widetilde{M}$ is a covering map $\pi : \widetilde{M} \to M$, whose group of deck transformations can be identified with $\Gamma = \{\, g \in G \mid \widetilde{M}g = \widetilde{M} \,\}$, obtaining a homomorphism $h : \pi_1(M) \to G$ whose image is Γ. It turns out that D is an h-equivariant fiber bundle whose fibers are the leaves of $\pi^*\mathcal{F}$.

Next, the particular case of TP foliations on compact manifolds is described by Molino as follows.

Theorem A.2 (Molino [33, 34]). *Let $\mathcal{F}$ be a TP foliation on a compact connected manifold M. Then there are:*

- *a fiber bundle $\pi : M \to W$; and*
- *a Lie algebra $\mathfrak{g}$;*

such that the fibers of π are the leaf closures of $\mathcal{F}$, and the restriction of $\mathcal{F}$ to each leaf closure is a $\mathfrak{g}$-Lie foliation.

To prove Theorem A.2, it is shown first a lemma stating that, if a transitive foliation $\mathcal{F}$ on a compact manifold M has closed leaves, then its leaves are the fibers of a fiber bundle $\pi : M \to W$. This lemma is proved by taking a finite dimensional linear subspace $V \subset \mathfrak{X}(M, \mathcal{F})$ such that the evaluation map $V \to T_xM$ is surjective for all $x \in M$; set $k = \dim V$. Given any leaf F of $\mathcal{F}$, let $\mathcal{G}$ be the foliation of $F \times \mathbb{R}^k$ whose leaves are the fibers of the second factor projection $F \times \mathbb{R}^k \to \mathbb{R}^k$. Let $X_1, \dots, X_k$ be the elements of a basis of V, and let $\phi_t^1, \dots, \phi_t^k$ denote the corresponding foliated flows of $(M, \mathcal{F})$. Then a foliated submersion $\Phi : (F \times \mathbb{R}^k, \mathcal{G}) \to (M, \mathcal{F})$, which restricts to diffeomorphisms between the leaves, is defined by $\Phi(x; t_1, \dots, t_k) = \phi^1_{t_1} \cdots \phi^k_{t_k}(x)$. It follows that there is some open neighborhood Ω of 0 in $\mathbb{R}^k$ and some saturated open neighborhood of F in M such that Φ restricts to a diffeomorphism $F \times \Omega \to U$; this is a fiber bundle trivialization that shows the lemma.

Now, when $\mathcal{F}$ is TP, it is proved that its leaf closures are the leaves of a transitive foliation $\overline{\mathcal{F}}$. Thus Theorem A.2 follows by applying the above lemma to $\overline{\mathcal{F}}$.

In Theorem A.2, it is said that π, W and $\mathfrak{g}$ are the *basic fibration*, *basic manifold* and *structural Lie algebra* of $\mathcal{F}$, respectively.

The final step to describe general Riemannian foliations on compact manifolds is the following.

Theorem A.3 (Molino [33, 34])**.** *Let $\mathcal{F}$ be a Riemannian foliation codimension q on a compact manifold M. Then there is an $\mathrm{O}(q)$-principal bundle $\hat{\pi} : \widehat{M} \to M$ equipped with an $\mathrm{O}(q)$-invariant TP foliation $\widehat{\mathcal{F}}$ such that $\hat{\pi}$ is a foliated map $(\widehat{M}, \widehat{\mathcal{F}}) \to (M, \mathcal{F})$, and its restrictions to the leaves of $\widehat{\mathcal{F}}$ are the holonomy covers of the corresponding leaves of $\mathcal{F}$.*

The proof of Theorem A.3 is easy. It follows by considering any $\nabla^{\mathcal{F}}$-flat Riemannian structure on $N\mathcal{F}$. Then $\mathcal{F}_P$ restricts to the $\mathrm{O}(q)$-principal bundle $\hat{\pi} : \widehat{M} \to M$ of orthonormal frames of $N\mathcal{F}$, obtaining a foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$. Moreover an easy adaptation of classical arguments show that $\widehat{\mathcal{F}}$ is TP.

A global descriptive picture of a Riemannian foliation $\mathcal{F}$ on a compact manifold M is obtained by successively applying Theorem A.3 to $\mathcal{F}$, Theorem A.2 to $\widehat{\mathcal{F}}$, and Theorem A.1 to the restriction of $\widehat{\mathcal{F}}$ to each of its leaf closures. In this way, many problems about Riemannian foliations can be reduced to the case of Lie foliations, or even to problems in Lie groups.

The structural Lie algebra of $\widehat{\mathcal{F}}$ is called *structural Lie algebra* of $\mathcal{F}$; this definition agrees with the above ones for TP or Lie foliations. Observe that, by Theorems A.3 and A.2, the space $M/\overline{\mathcal{F}}$ of leaf closures of $\mathcal{F}$ is homeomorphic to the orbit space $W/\mathrm{O}(q)$, where W is the basic manifold of $\widehat{\mathcal{F}}$.

Remark 33. The above descriptive theorems can be stated for foliations on open manifolds as far as the transverse parallelisms involved can be represented by complete vector fields; this gives rise to the concept of *transversely complete* Riemannian foliation, which can be used as hypothesis instead of the compactness of M.

A.5 Structural transverse action of a Lie foliation

Here, we recall from [7] another characteristic property of Lie foliations.

For any Lie algebra $\mathfrak{g}$, a homomorphism $\mathfrak{g} \to \overline{\mathfrak{X}}(M, \mathcal{F})$ is called an *infinitesimal transverse action* of $\mathfrak{g}$ on $(M, \mathcal{F})$. In particular, we always have a canonical infinitesimal transverse action of $\overline{\mathfrak{X}}(M, \mathcal{F})$ on $(M, \mathcal{F})$. An infinitesimal transverse action $\theta : \mathfrak{g} \to \overline{\mathfrak{X}}(M, \mathcal{F})$ is called:

- *faithful* if it is injective; and
- *transitive* if the composition

$$\mathfrak{g} \xrightarrow{\theta} \overline{\mathfrak{X}}(M, \mathcal{F}) \xrightarrow{\overline{\mathrm{ev}}_x} N_x\mathcal{F}$$

  is surjective for all $x \in M$.

For any group G, an anti-homomorphism $\Phi : G \to \overline{\mathrm{Diffeo}}(M, \mathcal{F})$, $g \mapsto \Phi_g$, is called a (*right*) *transverse action* of G on $(M, \mathcal{F})$. A transverse action of G on $(M, \mathcal{F})$ is called:

- *faithful* if it is injective; and
- *transitive* if the induced action of $\overline{\mathrm{Diffeo}}(M, \mathcal{F})$ on $M/\mathcal{F}$ is transitive.

For an open subset $O \subset G$, a map $\phi : M \times O \to M$ is called a *local representation* of Φ on O if $\phi_g = \phi(\cdot, g) \in \Phi_g$ for all $g \in O$. When G is a Lie group, Φ is said to be *smooth* if it has a smooth local representation around each element of G.

If G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, then there is a canonical bijection between infinitesimal transverse actions of $\mathfrak{g}$ on $(M, \mathcal{F})$ and smooth transverse actions of G on $(M, \mathcal{F})$ so that the infinitesimal transverse action $\theta : \mathfrak{g} \to \overline{\mathfrak{X}}(M, \mathcal{F})$ corresponding to a transverse action $\Phi : G \to \overline{\mathrm{Diffeo}}(M, \mathcal{F})$ is determined by the commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\phi^x_*"] \arrow[d, "\theta"'] & T_xM \arrow[d, "\text{projection}"] \\
\overline{\mathfrak{X}}(M, \mathcal{F}) \arrow[r, "\overline{\mathrm{ev}}_x"] & N_x\mathcal{F}
\end{tikzcd}$$

for all $x \in M$, where $\phi^x = \phi(x, \cdot)$ for any smooth local representation ϕ of Φ around the identity element e, and $\mathfrak{g}$ is identified with T_eG. Moreover faithful/transitive transverse actions correspond to faithful/transitive infinitesimal transverse actions by this correspondence.

Remark 34. Similarly, *left transverse actions* of G on $(M, \mathcal{F})$ can be defined as homomorphisms $G \to \overline{\mathrm{Diffeo}}(M, \mathcal{F})$, and correspond to infinitesimal transverse actions of $\mathfrak{g}^-$ on $(M, \mathcal{F})$.

With the above terminology, by definition, $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation just when there is a faithful transitive transverse action of $\mathfrak{g}$ on $(M, \mathcal{F})$. Hence another characteristic property of Lie foliations is the existence of a smooth transverse action Φ of G on $(M, \mathcal{F})$ such that, for some smooth local representation ϕ of Φ around e, the composite

$$T_eG \xrightarrow{\phi_*^x} T_xM \xrightarrow{\text{projection}} N_x\mathcal{F}$$

is an isomorphism for all $x \in M$. This condition is independent of the choice of ϕ. This Φ is called the *structural transverse action* of $\mathcal{F}$.

A.6 Growth of Riemannian foliations

There are many consequences of Molino's theory, with very different flavors: classification in particular cases, growth, cohomology, tautness and tenseness, and global analysis. Here, we recall the consequences about growth of Riemannian foliations. The study of the growth of the leaves of Riemannian foliations was begun by Carrière [13], showing the following result.

Theorem A.4 (Carrière [13]). *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M with structural Lie algebra $\mathfrak{g}$. Then:*

(*i*) *the holonomy covers of the leaves of $\mathcal{F}$ are Følner if and only if $\mathfrak{g}$ is solvable;*

(*ii*) *the holonomy covers of the leaves of $\mathcal{F}$ have polynomial growth if and only if $\mathfrak{g}$ is nilpotent; and,*

(*iii*) *in the case* (*ii*)*, the holonomy covers of the leaves have polynomial growth with degree less or equal than the nilpotence degree of $\mathfrak{g}$.*

By using Molino's structure theorems (Theorems A.2 and A.3), the proof of Theorem A.4 can be reduced to the case of Lie foliations with dense leaves. In this special case, Fedida's description (Theorem A.1) is used to obtain the following. If G and Γ denote the structurel Lie group and holonomy subgroup, then the holonomy pseudogroup is represented by the pseudogroup $\mathcal{H}$ on G generated by the left translations by elements of Γ, and moreover the growth of the leaves is equal to the so called *local growth* of Γ in G, which is defined as the growth of the orbits of the restriction of $\mathcal{H}$ to some neighborhood of the identity in G. In this way, Theorem A.4 follows by studying the local growth of an arbitrary dense finitely generated subgroup Γ of a Lie group G; this study is very delicate and involved.

Carrière also asked in [13] about the existence of a Riemannian foliation on a compact manifold such that the holonomy covers of the leaves have neither polynomial nor exponential growth. A negative answer to this question was recently given by Breuillard-Gelander [11], obtaining the following dichotomy.

Theorem A.5 (Breuillard-Gelander [11, Theorem 10.1])**.** *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. Then:*

- *either all holonomy covers of the leaves of $\mathcal{F}$ have polynomial growth with degree bounded by a common constant;*
- *or all holonomy covers of the leaves of $\mathcal{F}$ have exponential growth.*

Again, the results of Molino and Fedida (Theorems A.1–A.3) are used by Breuillard-Gelander to reduce the proof of Theorem A.5 to the following property of an arbitrary dense finitely generated group Γ in a Lie group G: if G is not nilpotent, then the local growth of Γ in G is exponential. This is a consequence of Proposition 3.11.1, which is part of their study of a topological Tits alternative.

B The pseudogroup $\widehat{\mathcal{H}}^{(2)}$

Assume that T and $\mathcal{H}$ satisfies the conditions of Section 4.1, and consider the notation of Chapter 4. Here, we construct a pseudogroup $\widehat{\mathcal{H}}^{(2)}$ on $\widehat{T}$, and a sub-pseudogroup $\widehat{\mathcal{H}} \subset \widehat{\mathcal{H}}^{(2)}$, naturally associated to $\mathcal{H}$. If $\mathcal{H}$ is minimal, then $\widehat{T}_0$ is $\widehat{\mathcal{H}}$-invariant, and the maps in $\widehat{\mathcal{H}}$ restrict to local transformations of $\widehat{T}_0$ that form $\widehat{\mathcal{H}}_0$.

Let $\mathcal{H}^{(2)}$ be the pseudogroup on $T^2 = T \times T$ generated by $S^2 = S \times S$; i.e., by the maps

$$h_2 \times h_1 : \operatorname{dom} h_2 \times \operatorname{dom} h_1 \to \operatorname{im} h_2 \times \operatorname{im} h_1$$

for $h_1, h_2 \in S$. Recall that s and t denote the source and target maps $\widehat{T} \to T$. For $h = h_2 \times h_1 \in S^2$, let

$$\hat{h} : t^{-1}(\operatorname{dom} h_2) \cap s^{-1}(\operatorname{dom} h_1) \to t^{-1}(\operatorname{im} h_2) \cap s^{-1}(\operatorname{im} h_1)$$

be defined by

$$\hat{h}(\boldsymbol{\gamma}(g, x)) = \boldsymbol{\gamma}(h_2 g h_1^{-1}, h_1(x)) ,$$

where $g \in \overline{S}$ and $x \in \operatorname{dom} g \cap \operatorname{dom} h_1$ with $g(x) \in \operatorname{dom} h_2$.

Lemma B.1. *If $O_1, O_2 \subset T$ are open with $\mathrm{id}_{O_1}, \mathrm{id}_{O_2} \in S$, then $\widehat{\mathrm{id}_{O_1 \times O_2}} = \mathrm{id}_{t^{-1}(O_1) \cap s^{-1}(O_2)}$.*

Proof. For $g \in \overline{S}$ and $x \in \operatorname{dom} g \cap O_1$ with $g(x) \in O_2$, we have

$$\widehat{\mathrm{id}_{O_1 \times O_2}}(\boldsymbol{\gamma}(g, x)) = \boldsymbol{\gamma}(\mathrm{id}_{O_2}\, g \,\mathrm{id}_{O_1}^{-1}, \mathrm{id}_{O_1}(x))) = \boldsymbol{\gamma}(g, x) . \quad \square$$

Lemma B.2. *For $h = h_2 \times h_1$ and $h' = h_2' \times h_1'$ in S^2, we have $\widehat{h'h} = \widehat{h'}\hat{h}$.*

Proof. We have

$$\begin{aligned}
\operatorname{dom}(\widehat{h'}\hat{h}) &= \widehat{h'}^{-1}(\operatorname{dom}\widehat{h'} \cap \operatorname{im}\hat{h}) \\
&= \widehat{h'}^{-1}\left(t^{-1}(\operatorname{dom}h_2') \cap s^{-1}(\operatorname{dom}h_1') \cap t^{-1}(\operatorname{im}h_2) \cap s^{-1}(\operatorname{im}h_1)\right) \\
&= \widehat{h'}^{-1}\left(t^{-1}(\operatorname{dom}h_2' \cap \operatorname{im}h_2) \cap s^{-1}(\operatorname{dom}h_1' \cap \operatorname{im}h_1)\right) \\
&= \{\, \boldsymbol{\gamma}(g,x) \mid g \in \overline{S},\ x \in \operatorname{dom}g,\ \boldsymbol{\gamma}(g,x) \in \operatorname{dom}\hat{h}, \\
&\qquad \hat{h}(\boldsymbol{\gamma}(g,x)) \in t^{-1}(\operatorname{dom}h_2' \cap \operatorname{im}h_2) \cap s^{-1}(\operatorname{dom}h_1' \cap \operatorname{im}h_1) \,\} \\
&= \{\, \boldsymbol{\gamma}(g,x) \mid g \in \overline{S},\ x \in \operatorname{dom}g \cap \operatorname{dom}h_1,\ g(x) \in \operatorname{dom}h_2, \\
&\qquad h_1(x) \in \operatorname{dom}h_1',\ h_2g(x) \in \operatorname{dom}h_2' \,\} \\
&= t^{-1}\left(h_2^{-1}(\operatorname{dom}h_2' \cap \operatorname{im}h_2)\right) \cap s^{-1}\left(h_1^{-1}(\operatorname{dom}h_1' \cap \operatorname{im}h_1)\right) \\
&= t^{-1}(\operatorname{dom}(h_2'h_2)) \cap s^{-1}(\operatorname{dom}(h_1'h_1)) \\
&= \operatorname{dom}\widehat{h'h}\ .
\end{aligned}$$

Now let $\boldsymbol{\gamma}(g,x) \in \operatorname{dom}(\widehat{h'}\hat{h}) = \operatorname{dom}\widehat{h'h}$; thus $g \in \overline{S}$, $x \in \operatorname{dom}g \cap \operatorname{dom}h_1$, $g(x) \in \operatorname{dom}h_2$, $h_1(x) \in \operatorname{dom}h_1'$, $h_2g(x) \in \operatorname{dom}h_2'$. Then

$$\begin{aligned}
\widehat{h'h}(\boldsymbol{\gamma}(g,x)) &= \boldsymbol{\gamma}(h_2'h_2g(h_1'h_1)^{-1}, h_1'h(x)) \\
&= \boldsymbol{\gamma}(h_2'h_2gh_1^{-1}(h_1')^{-1}, h_1'h(x)) \\
&= \widehat{h'}(\boldsymbol{\gamma}(h_2gh_1^{-1}, h_1(x))) \\
&= \widehat{h'}\hat{h}(\boldsymbol{\gamma}(g,x))
\end{aligned}$$

because

$$h'h = (h_2' \times h_1')(h_2 \times h_1) = h_2'h_2 \times h_1'h_1. \quad \square$$

Corollary B.3. *For $h = h_2 \times h_1 \in S^2$, the map $\hat{h}$ is bijective with $\hat{h}^{-1} = \widehat{h^{-1}}$.*

Proof. By Lemmas B.1 and B.2, we have

$$\widehat{h^{-1}}\hat{h} = \widehat{h^{-1}h} = \widehat{\operatorname{id}_{\operatorname{dom}h}} = \operatorname{id}_{s^{-1}(\operatorname{dom}h_1)\cap t^{-1}(\operatorname{dom}h_2)} = \operatorname{id}_{\operatorname{dom}\hat{h}}\ . \quad \square$$

Lemma B.4. *$\hat{h}$ is a homeomorphism for all $h = h_2' \times h_1' \in S^2$.*

Proof. By Corollary B.3, it is enough to prove that $\hat{h}$ is continuous, which holds because it can be expressed as the composition of the following continuous maps:

$$
\begin{aligned}
&t^{-1}(\operatorname{dom} h_2) \cap s^{-1}(\operatorname{dom} h_1) \\
&\qquad \xrightarrow{(\mathrm{const}_{h_2}, t, \mathrm{id}, \mathrm{const}_{h_1^{-1}}, h_1 s)} \{h_2\} \times \operatorname{dom} h_2 \times (t^{-1}(\operatorname{dom} h_2) \cap s^{-1}(\operatorname{dom} h_1)) \\
&\qquad\qquad\qquad\qquad\qquad \times \{h_1^{-1}\} \times \operatorname{im} h_1 \\
&\qquad \xrightarrow{\boldsymbol{\gamma} \times \mathrm{id} \times \boldsymbol{\gamma}} \boldsymbol{\gamma}(\{h_2\} \times \operatorname{dom} h_2) \times (t^{-1}(\operatorname{dom} h_2) \cap s^{-1}(\operatorname{dom} h_1)) \\
&\qquad\qquad \times \boldsymbol{\gamma}(\{h^{-1}\} \times \operatorname{im} h_1) \\
&\qquad \xrightarrow{\text{product}} t^{-1}(\operatorname{im} h_2) \cap s^{-1}(\operatorname{im} h_1)\ ,
\end{aligned}
$$

as can be checked on elements:

$$
\begin{aligned}
\boldsymbol{\gamma}(g,x) &\mapsto (h_2, g(x), \boldsymbol{\gamma}(g,x), h_1^{-1}, h_1(x)) \\
&\mapsto (\boldsymbol{\gamma}(h_2, g(x)), \boldsymbol{\gamma}(g,x), (\boldsymbol{\gamma}(h_1^{-1}, h_1(x))) \\
&\mapsto \boldsymbol{\gamma}(h_2 g h_1^{-1}, h_1(x)) = \hat{h}(\boldsymbol{\gamma}(g,x))\ . \quad \square
\end{aligned}
$$

Let $\widehat{\mathcal{H}}^{(2)}$ be the pseudogroup generated by $\widehat{S^2} = \{\, \hat{h} \mid h \in S^2 \,\}$. Lemma B.2 and Corollary B.3 give the following.

Corollary B.5. *$\widehat{S^2}$ is a pseudo∗group.*

Since $S^1 = \{h_1 \times \mathrm{id}_T \mid h_1 \in S\,\} \equiv S$ is a sub-pseudo∗group of S^2, we get that $\widehat{S^1} = \{\, \hat{h} \mid h \in S^1 \,\}$ is a sub-pseudo∗group of $\widehat{S^2}$. Let $\widehat{\mathcal{H}} \subset \widehat{\mathcal{H}}^{(2)}$ be the sub-pseudogroup generated by $\widehat{S^1}$. Observe that, if $\mathcal{H}$ is minimal, then the subspace $\widehat{T}_0$, defined with any given $x_0 \in T$, is $\widehat{\mathcal{H}}$-invariant, and the maps in $\widehat{\mathcal{H}}$ restrict to local transformations of $\widehat{T}_0$ that form $\widehat{\mathcal{H}}_0$.

Lemma B.6. *$\widehat{T}_U$ meets all the orbits of $\widehat{\mathcal{H}}^{(2)}$.*

Proof. Let $\boldsymbol{\gamma}(g,x) \in \widehat{T}$ with $g \in \overline{S}$ and $x \in \operatorname{dom} g$. Since U meets all orbits of $\mathcal{H}$, there are some $h_1, h_2 \in S$ such that $x \in \operatorname{dom} h_1$, $g(x) \in \operatorname{dom} h_2$, $h_1^{-1}(x) \in U$ and $h_2 g(x) \in U$. Then $\boldsymbol{\gamma}(g,x) \in \operatorname{dom} \hat{h}$ for $h = h_2 \times h_1 \in S^2$, and $\hat{h}(\boldsymbol{\gamma}(g,x)) = \boldsymbol{\gamma}(h_2 g h_1^{-1}, h_1(x))$ satisfies

$$
s(\hat{h}(\boldsymbol{\gamma}(g,x))) = h_1(x) \in U\ , \quad t(\hat{h}(\boldsymbol{\gamma}(g,x))) = h_2 g(x) \in U\ .
$$

Hence

$$
\hat{h}(\boldsymbol{\gamma}(g,x)) \in s^{-1}(U) \cap t^{-1}(U) = \widehat{T}_U\ . \quad \square
$$

Since the compact generation of $\mathcal{H}$ is satisfied with U, there is a symmetric set of generators, $\{f_1, \dots, f_m\}$, of $\mathcal{H}|_U$, which can be choosen in S, such that each f_a has an extension $\tilde{f}_a \in \mathcal{H}$ with $\overline{\operatorname{dom} f_a} \subset \operatorname{dom} \tilde{f}_a$. Set also $f_0 = \mathrm{id}_U$ and $\tilde{f}_0 = \mathrm{id}_T$.

Lemma B.7. *The maps $f_{ab} = f_a \times f_b$ $(a, b \in \{0, \dots, m\})$ generate $\mathcal{H}^{(2)}|_{U\times U}$, and each $\tilde{f}_{ab} = \tilde{f}_a \times \tilde{f}_a \in \mathcal{H}^{(2)}$ is an extension of f_{ab} with $\overline{\operatorname{dom} f_{ab}} \subset \operatorname{dom} \tilde{f}_{ab}$; in particular, $\mathcal{H}^{(2)}$ is compactly generated.*

Proof. Since $\mathcal{H}^{(2)}|_{U\times U}$ is generated by $\mathcal{H}|_U \times \mathcal{H}|_U \subset \mathcal{H}^2$, it is enough to prove that any $h_2 \times h_1 \in \mathcal{H}|_U \times \mathcal{H}|_U$ can be written as a composition of maps f_{ab} around any point $(x_2, x_1) \in \operatorname{dom} h_2 \times \operatorname{dom} h_1$. Because $\{f_1, \dots, f_m\}$ generates $\mathcal{H}|_U$, we have $h_1 = f_{a_p} \cdots f_{a_1}$ around x_1 and $h_2 = f_{b_q} \cdots f_{b_1}$ around x_2 for some $a_1, \dots, a_p, b_1, \dots, b_q \in \{1, \dots, m\}$. By composing with $f_0 = \mathrm{id}_U$ if needed, we can assume that $p = q$ by allowing the possibility that some of the indices $a_1, \dots, a_p, b_1, \dots, b_q$ may be 0. Then

$$h_2 \times h_1 = (f_{a_p} \times f_{b_p}) \cdots (f_{a_1} \times f_{b_1}) = f_{a_p b_p} \cdots f_{a_1 b_1}$$

around (x_2, x_1).

On the other hand, each $\tilde{f}_{ab} = \tilde{f}_a \times \tilde{f}_a \in \mathcal{H}^{(2)}$ is clearly an extension of f_{ab}, and we have

$$\overline{\operatorname{dom} f_{ab}} = \overline{\operatorname{dom} f_a} \times \overline{\operatorname{dom} f_b} \subset \operatorname{dom} \tilde{f}_a \times \operatorname{dom} \tilde{f}_b = \operatorname{dom} \tilde{f}_{ab} \,. \quad \square$$

Let $\widehat{\mathcal{H}}^{(2)}_U = \widehat{\mathcal{H}}^{(2)}|_{\widehat{T}_U}$.

Lemma B.8. *The maps $\widehat{f_{ab}}$ $(a, b \in \{0, \dots, m\})$ generate $\widehat{\mathcal{H}}^{(2)}_U$, and each $\widehat{\tilde{f}_{ab}} \in \widehat{\mathcal{H}}^{(2)}$ is an extension of $\widehat{f_{ab}}$ with $\overline{\widehat{f_{ab}}} \subset \operatorname{dom} \widehat{\tilde{f}_{ab}}$; in particular, $\widehat{\mathcal{H}}^{(2)}$ is compactly generated.*

Proof. Each $\widehat{\tilde{f}_{ab}}$ is an extenstion of $\widehat{f_{ab}}$ because $\tilde{f}_{ab}$ extends f_{ab} (Lemma B.7). Moreover

$$\begin{aligned} \overline{\operatorname{dom} \widehat{f_{ab}}} &= \overline{t^{-1}(\operatorname{dom} f_a) \cap s^{-1}(\operatorname{dom} f_b)} \\ &\subset t^{-1}(\overline{\operatorname{dom} f_a}) \cap s^{-1}(\overline{\operatorname{dom} f_b}) \\ &\subset t^{-1}(\operatorname{dom} \tilde{f}_a) \cap s^{-1}(\operatorname{dom} \tilde{f}_b) \\ &= \operatorname{dom} \widehat{\tilde{f}_{ab}} \,. \end{aligned}$$

Since the maps f_{ab} $(a, b \in \{0, \dots, m\})$ generate $\mathcal{H}^{(2)}|_{U\times U}$ (Lemma B.7), the corresponding maps $\widehat{f_{ab}}$ generate $\widehat{\mathcal{H}}^{(2)}_U$ by Lemmas B.1 and B.2, and Corollary B.3. $\square$

Corollary B.9. *$\widehat{\mathcal{H}}^{(2)}$ is compactly generated.*

Proof. The result follows from Corollary 4.3.10 and Lemmas B.6 and B.8. $\square$

Recall that the sets T_{i_k} form a finite covering of $\overline{U}$ by open subsets of T. Let $\{W'_k\}$ be a shrinking of $\{T_{i_k}\}$ as cover of $\overline{U}$ by open subsets of T. By applying Proposition 3.5.4 several times, we obtain finite covers, $\{W_p\}$, $\{V_a\}$ and $\{V'_u\}$, of $\overline{U}$ by open subsets of T, and corresponding shrinkings, $\{W_{0,k}\}$ of $\{W_k\}$ and $\{V'_{0,a}\}$ of $\{V_a\}$, as covers of $\overline{U}$ by open subsets of T, such that the following properties hold:

- For indices p, k and l, and all $h \in \mathcal{H}$ and $x \in \operatorname{dom} h \cap U \cap W_p \cap \cap W'_{0,k}$ with $h(x) \in U \cap W'_{0,l}$, there is some $\tilde{h} \in S$ such that

$$\overline{W_p} \subset \operatorname{dom} \tilde{h} \cap W'_k \ , \quad \boldsymbol{\gamma}(\tilde{h}, x) = \boldsymbol{\gamma}(h, x) \ , \quad \tilde{h}(\overline{V_a}) \subset W'_l \ .$$

- For indices a, p and q, and all $h \in \mathcal{H}$ and $x \in \operatorname{dom} h \cap U \cap V_a \cap W_{0,p}$ with $h(x) \in U \cap W_{0,q}$, there is some $\tilde{h} \in S$ such that

$$\overline{V_a} \subset \operatorname{dom} \tilde{h} \cap W_p \ , \quad \boldsymbol{\gamma}(\tilde{h}, x) = \boldsymbol{\gamma}(h, x) \ , \quad \tilde{h}(\overline{V_a}) \subset W_q \ .$$

- For indices u, a and b, and all $h \in \mathcal{H}$ and $x \in \operatorname{dom} h \cap U \cap V'_u \cap V_{0,a}$ with $h(x) \in U \cap V_{0,b}$, there is some $\tilde{h} \in S$ such that

$$\overline{V'_u} \subset \operatorname{dom} \tilde{h} \cap V_a \ , \quad \boldsymbol{\gamma}(\tilde{h}, x) = \boldsymbol{\gamma}(h, x) \ , \quad \tilde{h}(\overline{V'_u}) \subset V_b \ .$$

By the definition of $\overline{\mathcal{H}}$ and $\overline{S}$, it follows that these properties also hold for all $h \in \overline{\mathcal{H}}$ with $\tilde{h} \in \overline{S}$.

For indices a and q, let $\overline{S}_{0,a,q}$, $\overline{S}_{1,a,q} \subset \overline{S}$ be defined like $\overline{S}_0$ and $\overline{S}_1$ in (4.1) and (4.2) by using V_a and W_q instead of V and W. Like in Section 4.3, let $\overline{\boldsymbol{\gamma}}$ denote the germ map defined on $C(\overline{V_a}, \overline{W_q}) \times \overline{V_a}$, and let $\mathcal{R}_{a,q} : \overline{S}_{1,a,q} \to C(\overline{V_a}, \overline{W_q})$ be the restriction map, $f \mapsto f|_{\overline{V_a}}$. Since the sets $V_{0,a} \cap V'_{0,v}$ form a cover of $\overline{U}$, we get that the sets $\widehat{T}_{a,q,u} = \boldsymbol{\gamma}(\overline{S}_{0,a,q} \times (V_{0,a} \cap V'_{0,u}))$ form a finite cover of $\overline{\widehat{T}_U}$ by open subsets of $\widehat{T}$. Let $\widehat{T}_{U,a,q,u} = \widehat{T}_U \cap \widehat{T}_{a,q,u}$. Then

$$\overline{\boldsymbol{\gamma}} : \mathcal{R}_{a,q}(\overline{S}_{1,a,q}) \times \overline{V_{0,a} \cap V'_{0,u}} \to \overline{\boldsymbol{\gamma}}(\mathcal{R}_{a,q}(\overline{S}_{1,a,q}) \times \overline{V_{0,a} \cap V'_{0,u}}) \tag{8.1}$$

is a homeomorphism by Corollary 4.3.12. Since $\overline{V_a}$ is compact, the compact-open topology on $\mathcal{R}_{a,q}(\overline{S}_{1,a,q})$ equals the topology induced by the supremun metric $d_{a,q,l}$ on $C(\overline{V_a}, \overline{W_q})$, defined with the metric d_{i_l} on T_{i_l} for any index l such that $\overline{W_q} \subset W'_l$ (recall that $\overline{W'_l} \subset T_{i_l}$). Take some indexes p and k such that $\overline{V_a} \subset W_p$ and $\overline{W_p} \subset W'_p$. Then the topology of $\mathcal{R}_{a,q}(\overline{S}_{1,a,q}) \times \overline{V_{0,a} \cap V'_{0,u}}$ is induced by the metric $d_{a,u,q,k,l}$ given by

$$d_{a,q,u,k,l}((g, y), (g', y')) = d_{i_k}(y, y') + d_{a,q,l}(g, g')$$

(recall that $\overline{W'_k} \subset T_{i_k}$). Let $\hat{d}_{a,q,u,k,l}$ be the metric on $\mathcal{R}_{a,q}(\overline{S}_{1,a,q}) \times \overline{V_{0,a} \cap V'_{0,u}}$ that correspondes to $d_{a,q,u,k,l}$ by the homeomorphism (8.1); it induces the topology of $\overline{\boldsymbol{\gamma}}(\mathcal{R}_{a,q}(\overline{S}_{1,a,q}) \times \overline{V_{0,a} \cap V'_{0,u}})$.

For any indices v and n, define $\overline{S}'_{0,v,n}$ and $\overline{S}'_{1,v,n}$ like $\overline{S}_{0,a,q}$ and $\overline{S}_{1,a,q}$, by usising V'_v and W'_n (also like $\overline{S}_0$ and $\overline{S}_1$ in (4.1) and (4.2)). Let $\mathcal{R}'_{v,n} : \overline{S}'_{1,v,n} \to C(\overline{V'_v}, \overline{W'_n})$ denote the restriction map. Again, the compact-open topology on $\mathcal{R}'_{v,n}(\overline{S}'_{1,v,n})$ equals the topology induced by the supremun metric $d'_{v,n}$ on $C(\overline{V'_v}, \overline{W'_n})$, defined with the metric d_{i_n} on T_{i_n} (recall that $\overline{W'_n} \subset T_{i_n}$). Take some indices b, r and m such that $\overline{V'_v} \subset V_b$ and $\overline{V_b} \subset W_r$ and $\overline{W_r} \subset W'_m$. We can suppose that $\overline{V'_{0,u}} \subset V_{0,b}$, and take an index s such that $\overline{W_s} \subset W'_n$. Then we can consider the restricion map $\mathcal{R}^{v,n}_{b,s} : C(\overline{V_b}, \overline{W_s}) \to C(\overline{V'_v}, \overline{W_n})$. Its restriction $\mathcal{R}^{v,n}_{b,s} : \overline{S}_{1,b,s} \to \overline{S}'_{1,v,n}$ is injective by Remark 30, and surjective by Remark 29. So it is a continuous bijection between compact Hausdorff spaces, obtaining that it is a homeomorphism. Then, by compactness, it is a uniform homeomorphism with repect to the supremum metrics $d_{b,s,n}$ and $d'_{v,n}$. So, since b, s, v, m and n run in finite families of indices, there is a mapping $\varepsilon \to \delta_1(\varepsilon) > 0$, for $\varepsilon > 0$, such that

$$d_{b,s,n}(f, f') < \delta_1(\varepsilon) \Longrightarrow d'_{v,n}(\mathcal{R}^{v,n}_{b.s}(f), \mathcal{R}^{v,n}_{b,s}(f')) < \varepsilon \tag{8.2}$$

for all $f, f' \in \overline{S}_{1,b,s}$.

Lemma B.10. *$\widehat{\mathcal{H}}^{(2)}_U$ satisfies the equicontinuity condition with $\widehat{S^2} \cap \widehat{\mathcal{H}}^{(2)}_U$ and the quasi-local metric represented by the family $\{(\widehat{T}_{U,a,q,u}, \hat{d}_{a,q,u,k,l})\}$.*

Proof. Fix any $\varepsilon > 0$. Let $h = h_2 \times h_1 \in S^2$, and take $\boldsymbol{\gamma}(g, y)$ and $\boldsymbol{\gamma}(g', y')$ in $\widehat{T}_{U,q,a,u} \cap \hat{h}^{-1}(\widehat{T}_{U,b,s,v} \cap \operatorname{im} \widehat{h})$, where $g, g' \in \overline{S}_{0,a,q}$ and $y, y' \in V_{0,a} \cap V'_{0,u}$. By Remark 29, we can assume $\operatorname{dom} h_1 = T_{i_k}$ and $\operatorname{dom} h_2 = T_{i_l}$. Then the elements

$$\begin{aligned} \hat{h}(\boldsymbol{\gamma}(g, y)) &= \boldsymbol{\gamma}(h_2 g h_1^{-1}, h_1(y)) \ , \\ \hat{h}(\boldsymbol{\gamma}(g', y')) &= \boldsymbol{\gamma}(h_2 g' h_1^{-1}, h_1(y')) \end{aligned}$$

belong to $\widehat{T}_{U,b,s,v}$, which means that $h_1(y), h_1(y') \in V_{0,b} \cap V'_{0,v}$ and there are $f, f' \in \overline{S}_{0,b,s}$ so that

$$\begin{aligned} \boldsymbol{\gamma}(f, h_1(y)) &= \boldsymbol{\gamma}(h_2 g h^{-1}, h_1(y)) \ , \\ \boldsymbol{\gamma}(f', h_1(y')) &= \boldsymbol{\gamma}(h_2 g' h^{-1}, h_1(y')) \ ; \end{aligned}$$

in particular, $\overline{V_b} \subset \operatorname{dom} f \cap \operatorname{dom} f'$. In fact, we can assume that $\operatorname{dom} f = \operatorname{dom} f' = T_{i_m}$.

Observe that $\operatorname{im} h_1$ and $\operatorname{im} h_2$ may not be incluided in T_{i_m} and T_{i_n}, $\operatorname{im} g$ and $\operatorname{im} g'$ may not be incluided in T_{i_l}, and $\operatorname{im} f$ and $\operatorname{im} f'$ may not be incluided in T_{i_n}.

Claim 8. $\overline{V'_v} \subset \operatorname{im} h_1$ and $h_1^{-1}(\overline{V'_v}) \subset V_a$.

By the assumptions on $\{V'_u\}$, since $h_1(y) \in U \cap V'_v \cap V_{0,b} \cap \operatorname{dom} h_1^{-1}$ and $h_1^{-1} h_1(y) = y \in U \cap V'_u \cap V_{0,a}$, there is some $\widetilde{h_1^{-1}} \in S$ such that

$$\overline{V'}_v \subset \operatorname{dom} \widetilde{h_1^{-1}} \cap V_b \ , \quad \widetilde{h_1^{-1}}(\overline{V'_v}) \subset V_a \ ,$$
$$\boldsymbol{\gamma}(\widetilde{h_1^{-1}}, h_1(y)) = \boldsymbol{\gamma}(h_1^{-1}, h_1(y)) \ ;$$

in fact, we can suppose that $\operatorname{dom} \widetilde{h_1^{-1}} = T_{i_m}$ by Remark 29. Then

$$\widetilde{h_1^{-1}}(\overline{V'_v}) \subset V_a \subset T_{i_k} = \operatorname{dom} h_1 \ ,$$

yielding $\overline{V'_v} \subset \operatorname{dom}(h_1 \widetilde{h_1^{-1}})$. Moreover $\boldsymbol{\gamma}(h_1\widetilde{h_1^{-1}}, h_1(y)) = \boldsymbol{\gamma}(\mathrm{id}_T, h_1(y))$. Therefore $h_1 \widetilde{h_1^{-1}} = \mathrm{id}_{\operatorname{dom}(h_1 \widetilde{h_1^{-1}})}$ because $h_1, \widetilde{h_1^{-1}} \in S$. So $h_1 \widetilde{h_1^{-1}} = \mathrm{id}_T$ on some neighborhood of $\overline{V'_v}$, and therefore $\overline{V'_v} \subset \operatorname{im} h_1$ and $h_1^{-1} = \widetilde{h_1^{-1}}$ on $\overline{V'_v}$. Thus $h_1^{-1}(\overline{V'_v}) = \widetilde{h_1^{-1}}(\overline{V'_v}) \subset V_a$, which shows Claim 8.

By Claim 8,

$$\overline{V_a} \subset \operatorname{dom}(h_2 g h_1^{-1}) \cap \operatorname{dom}(h_2 g' h_1^{-1}) \ . \tag{8.3}$$

Since $f, f' \in \overline{S}_{0,b,s}$, we have $\overline{V_b} \subset \operatorname{dom} f \cap \operatorname{dom} f'$ and $f(\overline{V_b}) \cup f'(\overline{V_b}) \subset W_s$. On the other hand, from

$$\boldsymbol{\gamma}(f, h_1(y)) = \boldsymbol{\gamma}(h_2 g h_1^{-1}, h_1(y)) \ ,$$
$$\boldsymbol{\gamma}(f', h_1(y')) = \boldsymbol{\gamma}(h_2 g' h_1^{-1}, h_1(y')) \ ,$$

it follows that

$$h_2 g(y) = f h_1(y) \ , \quad h_2 g'(y') = f' h_1(y') \ ,$$
$$\boldsymbol{\gamma}(h_2 g h_1^{-1} f^{-1}, h_2 g(y)) = \boldsymbol{\gamma}(\mathrm{id}_T, h_2 g(y)) \ ,$$
$$\boldsymbol{\gamma}(h_2 g' h_1^{-1} f'^{-1}, h_2 g(y)) = \boldsymbol{\gamma}(\mathrm{id}_T, h_2 g'(y)) \ .$$

So, by Remark 30,

$$f(\overline{V'_v}) \subset \operatorname{dom}(h_2 g (h_1)^{-1} f^{-1}) \ , \quad f'(\overline{V'_v}) \subset \operatorname{dom}(h_2 g' h_1^{-1} f'^{-1}) \ ,$$

$h_2 g h_1^{-1} f^{-1} = \mathrm{id}_T$ in some neighborhood of $f(\overline{V'_v})$, and $h_2 g' h_1^{-1} f'^{-1} = \mathrm{id}_T$ in some neighborhood of $f'(\overline{V'_v})$. Thus $h_2 g h_1^{-1} = f$ and $h_2 g' h_1^{-1} = f'$ on some neighborhood of $\overline{V'_v}$; in particular,

$$\mathcal{R}^{v,n}_{b,s}(f) = h_2 g h_1^{-1}|_{\overline{V'_v}} \ , \quad \mathcal{R}^{v,n}_{b,s}(f') = h_2 g' h_1^{-1}|_{\overline{V'_v}} \ .$$

Consider the mappings $\varepsilon \mapsto \delta(\varepsilon) > 0$ and $\varepsilon \mapsto \delta_1(\varepsilon) > 0$ satisfying Remark 27 and (8.2). Then, for each $\varepsilon > 0$, define

$$\hat{\delta}(\varepsilon) = \min\{\delta(\varepsilon/2), \delta_1(\varepsilon/2)\} \ .$$

Given any $\varepsilon > 0$, suppose that

$$\hat{d}(\boldsymbol{\gamma}(g,y),\boldsymbol{\gamma}(g',y')) < \hat{\delta}(\varepsilon) \ .$$

This means that

$$\hat{d}((\mathcal{R}_a(g),y),(\mathcal{R}_a(g'),y')) < \hat{\delta}(\varepsilon) \ ,$$

or, equivalently,

$$d_{i_k}(y,y') + \sup_{x\in\overline{V_a}} d_{i_k}(g(y),g(y')) < \hat{\delta}(\varepsilon) \ .$$

Therefore

$$d_{i_k}(y,y') < \delta(\varepsilon/2) \ , \tag{8.4}$$

$$\sup_{x\in\overline{V_a}} d_{i_{k_0}}(g(x),g(x')) < \delta_1(\varepsilon/2) \ . \tag{8.5}$$

From (8.4) and Remark 27, it follows that

$$d_{i_l}(h(y),h(y')) < \varepsilon/2 \tag{8.6}$$

since $h \in S \subset \overline{S}$ and $y, y' \in T_{i_k} \cap h^{-1}(T_{i_k} \cap \operatorname{im} h)$. On the other hand, by Claim 8 and (8.5), we get

$$\begin{aligned}
d'_v(\mathcal{R}^v_b\mathcal{R}_b(f),\mathcal{R}^v_b\mathcal{R}_b(f')) &= \sup_{z\in\overline{V'_v}}\{d_{i_{k_0}}(gh^{-1}(z),g'h^{-1}(z))\} \\
&= \sup_{z\in h^{-1}(\overline{V'_v})} d_{i_{k_0}}(g(x),g'(x)) \\
&\leq \sup_{x\in\overline{V_a}} d_{i_{k_0}}(g(x),g'(x)) \\
&= d_a(\mathcal{R}_a(g),\mathcal{R}_a(g')) \\
&< \delta_1(\varepsilon/2).
\end{aligned}$$

So, by (8.2),

$$d_b(\mathcal{R}_b(f),\mathcal{R}_b(f')) < \varepsilon/2 \ . \tag{8.7}$$

From (8.6) and (8.7), we get

$$\begin{aligned}
&\hat{d}_{0,b,v,l}(\hat{h}(\boldsymbol{\gamma}(g,y)),\hat{h}(\boldsymbol{\gamma}(g',y'))) \\
&\qquad = \hat{d}_{0,b,v,l}(\boldsymbol{\gamma}(f,h(y)),\boldsymbol{\gamma}(f',h(y'))) \\
&\qquad = d_{b,v,l}((\mathcal{R}_b(f),h(y)),(\mathcal{R}_b(f'),h(y'))) \\
&\qquad = d_{i_l}(h(y),h(y')) + d_b(\mathcal{R}_b(f),\mathcal{R}_b(f')) < \varepsilon \ . \quad \square
\end{aligned}$$

Corollary B.11. *$\widehat{\mathcal{H}}^{(2)}$ is equicontinuous.*

Proof. $\widehat{\mathcal{H}}^{(2)}$ is equivalent to $\widehat{\mathcal{H}}^{(2)}_U$ by Lemma B.6. Thus the result follows from Lemma B.10 because the equicontinuity is preserved by equivalences. $\square$

Chapter 9
Conclusion

The main achievement of this thesis is a version of Molino's theory of Riemannian foliations in the topological context; i.e., for compact minimal equicontinuous foliated spaces. A condition of quasi-analyticity of the closure of the holonomy psudogroup was also used, which always holds for Riemannian foliations.

Molino's theory for Riemannian foliations has been an important tool, which was used in the proof of many results about this type of foliations. Similarly, our topological version of Molino's theory should have analogous applications. In particular, we have applied it in this thesis to study the growth of the leaves, obtaining a version of the corresponding result for Riemannian foliations.

The results about growth can be possibly improved for special types of structural local groups.

As another example of possible application, we hope that our theory will be also useful to study the leafwise heat flow on differential forms, which should preserve transverse continuity at infinite time under our assumptions.

The study of the basic cohomology could be possible for our foliated spaces thanks to our main theorem, like in the Riemannian foliation case. In the topological setting, the de Rham version of the basic cohomology is not possible. One should consider a version of singular cohomology, or perhaps other cohomologies better adapted to each special transverse model.

The special study of our description on particular examples is very interesting. For instance, our main theorem could give relevant information for foliated spaces given by: solenoids of arbitrary dimension, almost periodic functions on Lie groups, and almost periodic locally aperiodic complete connected Riemannian manifolds.

Our theory could also have implications in usual foliation theory, since the minimal sets are foliated spaces that may satisfy our assumptions.

Chapter 10
Resumen

Este trabajo trata sobre espacios foliados equicontinuos, considerados como generalizaciones de las foliaciones riemannianas introducidas por Reinhart [40]. Especialmente, se consideran espacios foliados equicontinuos compactos que son minimales en el sentido de que sus hojas son densas.

Fue indicado por Ghys en [34, Appendix E] (véase también el artículo [28] de Kellum) que los espacios foliados equicontinuos deberían considerarse como "foliaciones riemannianas topológicas", y por tanto muchos de los resultados sobre foliaciones riemannianas deberían tener versiones para espacios foliados equicontinuos. Algunos pasos en esa dirección fueron dados por Álvarez y Candel [4, 5], mostrando que, bajo condiciones razonables para espacios foliados equicontinuos compactos, las clausuras de sus hojas son espacios foliados minimales, sus hojas genéricas son quasi-isométricas entre sí, y su pseudogrupo de holonomía tiene una clausura, como en el caso de foliaciones riemannianas. En esta misma dirección, Matsumoto [30] probó que cualquier espacio foliado equicontinuo minimal tiene una medida transversa invariante no trivial, que es única salvo productos por constantes. La magnitud de la generalización de foliaciones riemannianas a espacios foliados equicontinuos fue precisada por Álvarez and Candel [5] (véase también el artículo [43] de Tarquini), dando una descripción topológica de las foliaciones riemannianas dentro de la clase de espacios foliados equicontinuos.

La mayoría de las propiedades de foliaciones riemannianas se siguen de una descripción debida a Molino [33, 34]. Sin embargo, hasta ahora, no había ninguna versión de la descripción de Molino para espacios foliados—las propiedades indicadas de espacios foliados equicontinuos fueron obtenidas por otros medios. El objetivo de nuestro trabajo es desarrollar tal versión de la teoría de Molino, y usarla para estudiar el crecimiento de las hojas en el mismo espíritu del estudio del crecimiento de foliaciones riemannianas por Carrière [13] (véase también el artículo reciente [11] por Breuillard-Gelander). Para entender mejor nuestros resultados, recordemos brevemente la teoría de Molino—una descripción más detallada de esa teoría se da en el Apéndice A.

Teoría de Molino para foliaciones riemannianas

Recuérdese que una *foliación* (diferenciable) $\mathcal{F}$ de *codimensión* q en una variedad M es una partición de M en subvariedades inmersas inyectivamente (*hojas*), que pueden ser descritas localmente por las fibras de submersiones locales en q-variedades. Estas submersiones, al igual que sus dominios, se dice que son *distinguidas*, y sus imágenes se llaman *cocientes locales*. Los cambios de submersiones distinguidas están dados por difeomorfismos entre subconjuntos abiertos de los cocientes locales, que se llaman *transformaciones de holonomía elementales*. Una foliación se llama *minimal* si las hojas son densas. Una aplicación entre variedades foliadas se denomina *foliada* si envía hojas a hojas.

Usando cadenas de abiertos distinguidos consecutivos a lo largo de lazos en una hoja L, y componiendo las correspondientes transformaciones de holonomía elementales, se consigue una representación de $\pi_1(L)$ en un grupo de gérmenes de esas composiciones, que se llama la *representación de holonomía* de L. Su imagen se denomina *grupo de holonomía* de L, y su núcleo es igual a la imagen de $\pi_1(\widetilde{L})$ para un único recubrimiento normal $\widetilde{L} \to L$, que se llama *recubrimiento de holonomía*. Para cualquier foliación general en una variedad segundo numerable, hay un subconjunto G_δ denso saturado cuyas hojas tienen grupos de holonomía triviales [26, 12]; así que un enunciado sobre recubrimientos de holonomía de las hojas se puede simplificar a un enunciado sobre hojas genéricas si se desea.

Sea $T\mathcal{F} \subset TM$ el subfibrado vectorial de vectores tangentes a las hojas. A $N\mathcal{F} = TM/T\mathcal{F}$ se le denomina *fibrado normal* de $\mathcal{F}$, y a sus secciones *campos de vectores normales*. Hay una conexión parcial plana a lo largo de las hojas en $N\mathcal{F}$ de forma que un campo de vectores normales local es paralelo a lo largo de las hojas si y sólo si es localmente proyectable por las submersiones distinguidas; términos como "plano a lo largo de las hojas," "paralelo a lo largo de las hojas" y "horizontal a lo largo de las hojas" se referirán a esta conexión parcial. Se dice que $\mathcal{F}$ es:

riemanniana si hay una estructura riemanniana paralela a lo largo de las hojas en $N\mathcal{F}$;

transitiva si el grupo de difeomorfismos foliados actúa transitivamente en M;

transversalmente paralelizable (TP) si existe una referencia global de $N\mathcal{F}$ que es paralela a lo largo de las hojas, llamada *paralelismo transverso*; y una

foliación de Lie si además el paralelismo transverso es una base de un álgebra de Lie con la operación inducida por el producto corchete de campos.

Estas condiciones son sucesivamente más fuertes. Intuitivamente, una foliación es riemanniana cuando sus hojas no se acercan ni alejan demasiado al desplazarse a lo largo de ellas.

La teoría de Molino describe las foliaciones riemannianas en variedades compactas en términos de foliaciones de Lie minimales, y usando las foliaciones TP como un paso intermedio:

1er **paso:** Si $\mathcal{F}$ es riemanniana y M compacta, entonces hay un $\mathrm{O}(q)$-fibrado principal, $\hat{\pi} : \widehat{M} \to M$, con una foliación TP $\mathrm{O}(q)$-invariante, $\widehat{\mathcal{F}}$, tal que $\hat{\pi}$ es una aplicación foliada cuyas restricciones a las hojas son los revestimientos de holonomía de las hojas de $\mathcal{F}$.

2$^{\text{o}}$ **paso:** Si $\mathcal{F}$ es TP y M compacta, entonces hay un fibrado $\pi : M \to W$ cuyas fibras son las clausuras de las hojas de $\mathcal{F}$, y la restricción de $\mathcal{F}$ a cada fibra es una foliación de Lie.

Las demostraciones de estos enunciados usan fuertemente la estructura diferencial de $\mathcal{F}$. En el 1er paso, $\hat{\pi} : \widehat{M} \to M$ es el $\mathrm{O}(q)$-fibrado principal de referencias ortonormales de $N\mathcal{F}$ respecto de alguna métrica paralela a lo largo de las hojas, y $\widehat{\mathcal{F}}$ está dada por la correspondiente conexión plana parcial a lo largo de las hojas. Entonces $\widehat{\mathcal{F}}$ es TP por la adaptación de un argumento estándar. En el 2$^{\text{o}}$ paso, se usan flujos foliados para producir trivializaciones de fibrado cuyas fibras sean las clausuras de las hojas; esto funciona ya que hay flujos foliados hacia cualquier dirección transversa por ser $\mathcal{F}$ TP.

Cuando $\mathcal{F}$ es minimal, se tiene lo siguiente:

Caso minimal: Si $\mathcal{F}$ es minimal y riemanniana, y M compacta, entonces, para algún subgrupo cerrado $H \subset \mathrm{O}(q)$, hay un H-fibrado principal, $\hat{\pi}_0 : \widehat{M}_0 \to M$, con una foliación de Lie minimal H-invariante, $\widehat{\mathcal{F}}_0$, tal que $\hat{\pi}_0$ es una aplicación foliada cuyas restricciones a las hojas son los recubrimientos de holonomía de las hojas de $\mathcal{F}$.

Esto se sigue de la combinación de ambos pasos al observar que cualquier clausura $\widehat{M}_0$ de una hoja de $\widehat{\mathcal{F}}$ es un subfibrado principal de $\hat{\pi} : \widehat{M} \to M$.

Una descripción útil de las foliaciones de Lie fue dada por Fédida [17, 18], pero no la consideraremos aquí.

Holonomía de las foliaciones riemannianas

Un *pseudogrupo* es una colección maximal de transformaciones locales de un espacio, que contiene la aplicación identidad, y es cerrada por las operaciones de composición, inversión, restricción y combinación (unión). Se puede considerar como un sistema dinámico generalizado, y todos los conceptos dinámicos básicos tienen versiones para pseudogrupos. Son relevantes en teoría de foliaciones porque las transformaciones de holonomía elementales generan un pseudogrupo que describe la dinámica transversa de $\mathcal{F}$; se denomina *pseudogrupo de holonomía* y sus elementos *transformaciones de holonomía*. Tal pseudogrupo está bien determinado salvo cierta *equivalencia* de pseudogrupos introducida por Haefliger [22, 23]. Se puede decir que $\mathcal{F}$ está *transversalmente modelado* por alguna clase de transformaciones locales si su pseudogrupo de holonomía se puede generar por ese tipo de transformaciones locales. Foliaciones riemannianas, TP y de Lie se pueden caracterizar respectivamente por estar transversalmente modelados por:

- isometrías locales de alguna variedad riemanniana;
- difeomorfismos locales de alguna variedad paralelizable que conservan el paralelismo; y
- traslaciones por la izquierda de algún grupo de Lie.

Así que foliaciones riemannianas son las transversalmente rígidas, y las foliaciones TP tienen un tipo de rigidez más restrictivo.

Cuando la variedad ambiente M es compacta, Haefliger [25] ha observado que el pseudogrupo de holonomía $\mathcal{H}$ de $\mathcal{F}$ satisface la siguiente propiedad:

Generación compacta: Hay algún subconjunto abierto relativamente compacto $U \subset T$, que corta todas las $\mathcal{H}$-órbitas, y hay un número finito de generadores $h_1, \dots, h_k$ de la restricción $\mathcal{H}|_U$ tal que cada h_i tiene extensión $\tilde{h} \in \mathcal{H}$ con $\overline{\operatorname{dom} h} \subset \operatorname{dom} \tilde{h}$.

Si además $\mathcal{F}$ es riemanniana, entonces Haefliger [23, 25] ha usado fuertemente también las siguientes propiedades de $\mathcal{H}$:

Completitud: Para todos los puntos $x, y \in T$, hay entornos abiertos, V de x y W de y, tal que, para todo $h \in \mathcal{H}$ y $z \in V \cap \operatorname{dom} h$ con $h(z) \in W$, hay algún $\tilde{h} \in \mathcal{H}$ tal que $\operatorname{dom} \tilde{h} = V$ y $\tilde{h} = h$ alrededor de z.

Clausura: Sea $J^1(T)$ el espacio de 1-jets de transformaciones locales de T, y sea $j^1(\mathcal{H}) \subset J^1(T)$ el subconjunto dado por 1-jets de aplicaciones en $\mathcal{H}$. Entonces la clausura $\overline{j^1(\mathcal{H})}$ en $J^1(T)$ es el conjunto de 1-jets de aplicaciones en un pseudogrupo $\overline{\mathcal{H}}$ de isometrías locales de T, llamado *clausura* de $\mathcal{H}$, cuyas órbitas son las clausuras de las órbitas de $\mathcal{H}$.

Cuasi-analiticidad: Si algún $h \in \mathcal{H}$ es la identidad en algún abierto O con $\overline{O} \subset \operatorname{dom} h$, entonces h es la identidad en algún entorno de $\overline{O}$.

La cuasi-analiticidad se cumple porque la diferencial de una isometría en un punto determina la aplicación en un entorno. Así que también la cumple $\overline{\mathcal{H}}$.

Para un pseudogrupo compactamente generado $\mathcal{H}$ de isometrías locales de una variedad riemanniana T, Salem ha dado una versión de la teoría de Molino ([42] y [34, Appendix D]; ver también [8]). En particular, en el caso minimal, resulta que hay un grupo de Lie G, un subgrupo compacto $K \subset G$ y un subgrupo denso finitamente generado $\Gamma \subset G$ tal que $\mathcal{H}$ es equivalente al pseudogrupo generado por la acción de Γ en el espacio homogénero G/K (esto fue observado también por Haefliger [23]).

Crecimiento de foliaciones riemannianas

La teoría de Molino tiene muchas consecuencias para una foliación riemanniana $\mathcal{F}$ en una variedad compacta M: sobre clasificación en casos particulares, crecimiento, cohomología, tensión y análisis global. En todas ellas, la teoría de Molino se usa para reducir el estudio al caso de foliaciones de Lie con hojas densas, donde normalmente se convierte en un problema de teoría de Lie. Una lista de referencias sobre todas las aplicaciones sería demasiado larga. Nos concentramos en las consecuencias sobre crecimiento de las hojas y sus recubrimientos de holonomía, lo que se refiere a su crecimiento como variedades riemannianas respecto de las métricas inducidas por cualquier métrica en M; este crecimiento depende sólo de $\mathcal{F}$ por la compacidad de M. Este estudio fue comenzado por Carrière [13], y recientemente continuado por Breuillard y Gelander, como consecuencia de su estudio de una alternativa de Tits topológica [11]. Sus resultados establecen lo siguiente, donde $\mathfrak{g}$ es el álgebra de Lie estructural de $\mathcal{F}$:

Teorema de Carrière: Los recubrimientos de holonomía de las hojas son:

- Følner si y sólo si $\mathfrak{g}$ es resoluble; y
- de crecimiento polinomial si y sólo si $\mathfrak{g}$ es nilpotente.

Además, en el segundo caso, el grado de su crecimiento polinomial está acotado por el grado de nilpotencia de $\mathfrak{g}$.

Teorema de Breuillard-Gelander: El crecimiento de los revestimientos de holonomía de todas las hojas es, o bien nilpotente, o bien exponencial.

Espacios foliados equicontinuos

Un *espacio foliado* $X \equiv (X, \mathcal{F})$ es un espacio topológico X dotado de una partición $\mathcal{F}$ en variedades conexas (*hojas*), que se pueden describir localmente por las fibras de submersiones locales topológicas. Se asumirá que X es localmente compacto y polaco. Un espacio foliado debe considerarse como una "foliación topológica". En este sentido, todas las nociones topológicas de foliaciones tienen versiones obvias para espacios foliados. En particular, el *pseudogrupo de holonomía* $\mathcal{H}$ de X se define en un espacio polaco localmente compacto T. Muchos resultados sobre foliaciones también tienen generalizaciones directas; por ejemplo, las hojas con grupo de holonomía trivial forman un conjunto G_δ denso, y $\mathcal{H}$ es compactamente generado si X es compacto. Incluso conceptos diferenciales en la dirección de las hojas se extienden fácilmente. Sin embargo esta tarea puede ser difícil o imposible para conceptos diferenciales transversos. Por ejemplo, el fibrado normal de un espacio foliado no tiene sentido en general: sería el espacio tangente de un espacio topológico en el caso de un espacio foliado por puntos. Así que el concepto de foliación riemanniana no se puede extender usando el fibrado normal; en vez de eso, esto se puede lograr usando el pseudogrupo de holonomía de la forma siguiente.

La rigidez transversa de una foliación riemanniana se puede traducir al espacio foliado X requiriendo equicontinuidad (uniforme) de $\mathcal{H}$; de hecho, la condición de equicontinuidad no es compatible con combinaciones de aplicaciones; así que la equicontinuidad se pide a algún subconjunto de generadores $S \subset \mathcal{H}$ que es cerrado por las operaciones de composición e inversión; tal S se llama un *pseudo∗grupo* con la terminología de Matsumoto [30]. Esto da lugar al concepto de espacio foliado *equicontinuo*, que se debe considerar como la versión topológica de foliación riemanniana.

Como en el caso de foliaciones riemannianas, Álvarez y Candel [4] han probado que, si el espacio foliado X es compacto y equicontinuo, las clausuras de sus órbitas son espacios foliados minimales, y su pseudogrupo de holonomía $\mathcal{H}$ es completo y tiene una clausura $\overline{\mathcal{H}}$. Con esta generalidad, $\overline{\mathcal{H}}$ no se puede definir usando 1-jets, por supuesto; en vez de eso, $\overline{\mathcal{H}}$ está formado por las aplicaciones que localmente son límites de aplicaciones de $\mathcal{H}$ con la topología compacto-abierta; este método funciona porque $\mathcal{H}$ es completo.

En el marco topológico, la cuasi-analiticidad de $\mathcal{H}$ (y $\overline{\mathcal{H}}$) no se sigue de la condición de equicontinuidad. Así que la cuasi-analiticidad se pedirá como hipótesis adicional cuando se necesite. De hecho, no funciona lo suficientemente bien cuando T no es localmente conexo. Por tanto se usará una propiedad llamada *equicontinuidad fuerte*, definida por la existencia de un pseudo∗grupo S, generando $\mathcal{H}$, tal que cualquier aplicación en S es la identidad en su dominio si lo es en algún subconjunto abierto no vacío; esta propiedad es más fuerte que la equicontinuidad sólo cuando T no es localmente conexo.

Las foliaciones transitivas y de Lie tienen las siguientes versiones topológicas obvias. Dado un grupo local G, se dice que el espacio foliado X es:

homogéneo si el grupo de sus transformaciones foliadas actúa transitivamente en X; y

G-espacio foliado si está transversalmente modelado por traslaciones locales a la izquierda en G.

Teoría de Molino topológica

El primer resultado principal de nuestro trabajo es la siguiente versión del caso minimal en la teoría de Molino.

Teorema A. *Sea $X \equiv (X, \mathcal{F})$ un espacio foliado polaco compacto, y $\mathcal{H}$ su pseudogrupo de holonomía. Supongamos que X es minimal y equicontinuo, y $\overline{\mathcal{H}}$ fuertemente quasi-analítico. Entonces hay un espacio foliado polaco compacto $\widehat{X}_0 \equiv (\widehat{X}_0, \widehat{\mathcal{F}}_0)$, una aplicación foliada $\hat{\pi}_0 : \widehat{X}_0 \to X$, y un grupo local G con una métrica invariante por la izquierda tal que:*

- *$\widehat{X}_0$ es un G-espacio foliado minimal;*
- *las fibras de $\hat{\pi}_0$ son homeomorfas entre sí; y*
- *las restricciones de $\hat{\pi}_0$ a las hojas son los revestimientos de holonomía de las hojas de $\mathcal{F}$.*

La principal dificultad para probar el Teorema A es que no hay fibrado normal de $\mathcal{F}$, mientras que $\widehat{X}_0$ se define como un subfibrado del fibrado de referencias ortonormales del fibrado normal en el caso de foliación riemanniana.

Para definir $\widehat{X}_0$, se construye primero lo que debería ser su pseudogrupo de holonomía $\widehat{\mathcal{H}}_0$ en un espacio $\widehat{T}_0$. Hasta cierto punto, esto ya había sido alcanzado por Álvarez and Candel [5], probando que, con las hipótesis del Teorema A, como en el caso de foliaciones, hay un grupo local G, un subgrupo compacto $K \subset G$ y un subgrupo local denso finitamente generado $\Gamma \subset G$ tal que $\mathcal{H}$ es equivalente al pseudogrupo generado por la acción local de Γ en G/K. Por consiguiente $\widehat{\mathcal{H}}_0$ debería ser equivalente al pseudogrupo generado por la acción local de Γ en G. Esto puede parecer un paso grande hacia la demostración, pero la realización de pseudogrupos compactamente generados como pseudogrupos de holonomía de espacios foliados compactos es imposible en general, según lo mostró Meigniez [32]. Esta dificultad se supera de la siguiente forma.

Consideramos un "buen" recubrimiento de X por abiertos distinguidos, $\{U_i\}$, con correspondientes submersiones distinguidas $p_i : U_i \to T_i$, y transformaciones de holonomía elementales $h_{ij} : T_{ij} \to T_{ji}$, donde $T_{ij} = p_i(U_i \cap U_j)$. Sea $\mathcal{H}$ el correspondiente representante del pseudogrupo de holonomía en

$T = \bigsqcup_i T_i$, generado por las aplicaciones h_{ij}. Entonces la construcción de $\widehat{\mathcal{H}}_0$ debe estar asociada a $\mathcal{H}$ de forma natural, para que sea inducido por un "buen" recubrimiento por abiertos distinguidos de algún espacio foliado compacto. En el caso de una foliación riemanniana, las buenas elecciones de $\widehat{T}_0$ y $\widehat{\mathcal{H}}_0$ son las siguientes:

- Consideramos una métrica riemanniana $\mathcal{H}$-invariante en T. Fijamos una referencia ortonormal $\hat{x}_0$ en algún punto x_0 de T. Entonces $\widehat{T}_0$ es la clausura de
$$\{\, h_*(\hat{x}_0) \mid h \in \mathcal{H},\ x_0 \in \operatorname{dom} h \,\}$$
en el fibrado de referencias ortonormales; así que
$$\widehat{T}_0 = \{\, g_*(\hat{x}_0) \mid g \in \overline{\mathcal{H}},\ x_0 \in \operatorname{dom} g \,\}\ . \qquad (*)$$

- $\widehat{\mathcal{H}}_0$ está generado por las diferenciales de aplicaciones en $\mathcal{H}$.

Estos conceptos diferenciales se pueden modificar de la siguiente forma:

- En $(*)$, cada $g_*(\hat{x}_0)$ determina el germen de g en x_0, $\boldsymbol{\gamma}(g, x_0)$, por la cuasi-analiticidad fuerte de $\overline{\mathcal{H}}$. Por tanto también determina $\boldsymbol{\gamma}(f, x)$, donde $f = g^{-1}$ y $x = g(x_0)$. Así que
$$\widehat{T}_0 \equiv \{\, \boldsymbol{\gamma}(f, x) \mid f \in \overline{\mathcal{H}},\ x \in \operatorname{dom} f,\ f(x) = x_0 \,\}\ . \qquad (**)$$

- La proyección $\hat{\pi}_0 : \widehat{T}_0 \to T$ corresponde via $(**)$ a la aplicación origen $\boldsymbol{\gamma}(f, x) \mapsto x$.

- Las diferenciales de aplicaciones $h \in \mathcal{H}$, actuando en referencias ortonormales, corresponden via $(**)$ a las aplicaciones $\hat{h}$ definidas por
$$\hat{h}(\boldsymbol{\gamma}(f, x)) = \boldsymbol{\gamma}(fh^{-1}, h(x))\ .$$

- La topología de $\widehat{T}_0$ se puede describir via $(**)$ de la siguiente manera. Sea $\overline{S}$ un pseudo$*$grupo generando $\overline{\mathcal{H}}$ y cumpliendo la condición de cuasi-analiticidad fuerte. Dotemos $\overline{S}$ con la topología compacto-abierta en aplicaciones parciales con dominios abiertos, según la define Abd-Allah-Brown [1], y consideramos el subespacio
$$\overline{S} * T = \{\, (f, x) \in \overline{S} \mid x \in \operatorname{dom} f\} \subset \overline{S} \times T\ .$$
Entonces la topología de $\widehat{T}_0$ corresponde via $(**)$ a la topología cociente por la aplicación gérmen $\boldsymbol{\gamma} : \overline{S} * T \to \boldsymbol{\gamma}(\overline{S} * T) \equiv \widehat{T}_0$, que es diferente de la topología de haz en gérmenes.

Este punto de vista (reemplazando referencias ortonormales por gérmenes) se puede traducir literalmente al ámbito de espacios foliados, obteniendo un psedogrupo $\widehat{\mathcal{H}}_0$ en $\widehat{T}_0$.

Ahora, se consideran triples (x, i, γ), donde $x \in U_i$, $\gamma \in \widehat{T}_{i,0} := \hat{\pi}_0^{-1}(T_i)$ y $p_i(x) = \hat{\pi}_0(\gamma)$. Declaramos $(x, i, \gamma) \sim (y, j, \delta)$ si $x = y$ y $\delta = \widehat{h_{ij}}(\gamma)$. Entonces $\widehat{X}_0$ se define como el espacio cociente correspondiente; la clase de equivalencia de cada triple (x, i, γ) se denota por $[x, i, \gamma]$. La estructura foliada $\widehat{\mathcal{F}}_0$ en $\widehat{X}_0$ se determina al requerir que, para cada índice i, los elementos del tipo $[x, i, \gamma]$ forman un abierto distinguido $\widehat{U}_{i,0}$, con submersión distinguida $\hat{p}_{i,0} : \widehat{U}_{i,0} \to \widehat{T}_{i,0}$ dada por $\hat{p}_{i,0}([x, i, \gamma]) = \gamma$. La proyección $\hat{\pi}_0 : \widehat{X}_0 \to X$ se define por $\hat{\pi}_0([x, i, \gamma]) = x$. Las propiedades enunciadas en el Teorema A se cumplen con estas definiciones.

Salvo homeomorphismos foliados (respectivamente, isomorfismos locales), $\widehat{X}_0$ (respectivamente, G) es independiente de las elecciones hechas. Entonces G se puede llamar *grupo local estructural* de $\mathcal{F}$.

Crecimiento de espacios foliados equicontinuos

Digamos que un grupo local G *se puede aproximar por grupos de Lie locales nilpotentes* si, en cualquier entorno de la identidad, existe una sucesión de subgrupos normales compactos F_n tales que $F_{n+1} \subset F_n$, $\bigcap_n F_n = \{e\}$ y G/F_n es un grupo de Lie local nilpotente. Nuestro segundo resultado principal, es la siguiente versión topológica débil de los teoremas anteriores de Carrière y Breuillard-Gelander.

Teorema B. *Sea X un espacio foliado cumpliendo las condiciones del Teorema A, y sea G su grupo local estructural. Entonces una de las siguientes propiedades se cumple:*

- *G se puede aproximar por grupos de Lie locales nilpotentes; o*
- *los revestimientos de holonomía de las hojas de X tienen crecimiento exponencial.*

Al igual que en el caso de foliaciones riemannianas, el Teorema A reduce la demostración del Teorema B al caso de G-espacios foliados, donde se convierte en un problema sobre grupos locales. Entonces, como cada grupo local se puede aproximar por grupos de Lie locales en el sentido anterior, el resultado se obtiene aplicando el tipo de argumentos de Breuillard-Gelander.

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